

SPRINGER BASIC SETS AND MODULAR SPRINGER CORRESPONDENCE FOR CLASSICAL TYPES

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ABSTRACT. We define the notion of basic set data for finite groups (building on the notion of basic set, but including an order on the irreducible characters as part of the structure), and we prove that the Springer correspondence provides basic set data for Weyl groups. Then we use this to determine explicitly the modular Springer correspondence for classical types (for representations in odd characteristic). In order to do so, we compare the order on bipartitions introduced by Dipper and James with the order induced by the Springer correspondence.

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1. INTRODUCTION

In [Spr76], Springer defined a correspondence which associates a pair consisting of a nilpotent orbit and a local system to each irreducible representation of a Weyl group. This correspondence was explicitly determined in all types [Spr76, HS77, Sho79, Sho80, AL82].

This construction was generalized by Lusztig to include all pairs [Lus84]. The generalized Springer correspondence was explicitly determined in [Lus84, LS85, Spa85, Lus86].

The Springer correspondence and its generalization are closely related to (generalized) Green functions. The orthogonality relations for generalized Green functions lead to an algorithm to compute the stalks of the intersection cohomology complexes of all nilpotent orbits, with coefficients in any equivariant local system.

In [Jut07], a modular Springer correspondence was defined, following the Fourier-Deligne approach of [Bry86]. Besides, one can see the decomposition matrix of a Weyl group as a submatrix of a decomposition matrix for equivariant perverse sheaves on the nilpotent cone defined in [Jut09]. The modular Springer correspondence was explicitly determined in [Jut07] in type A_n , and also in other types of rank at most three. These results are now in [Jut], together with the explicit determination of the modular Springer correspondence for exceptional Weyl groups.

In the present paper, we determine the modular Springer correspondence for classical types when the characteristic ℓ is different from 2. Observe that the case $\ell = 2$ was solved in [?, Corollary 9.8]. In complement with our results and [Jut], the modular Springer correspondence is therefore completely determined in all cases.

Our strategy is to use the known results in characteristic zero, and unitriangularity properties of the decomposition matrices (both for the Weyl groups and for the perverse sheaves) so that there is only one possibility for the correspondence.

The unitriangularity properties of the decomposition matrices are related to the notion of *basic sets*, which are subsets of the set of ordinary irreducible characters. Usually an order on the characters is fixed. We define the more precise notion of *basic set datum*, where the order is the first half of the structure, the other half being an injection from the modular characters to the ordinary characters. This is because we will consider several orders at the same time. The definition and first properties of basic set data are the subject of Section 2.

In Section 3, we will see that the Springer correspondence gives rise to basic set data for Weyl groups. The situation is much simpler when the

characteristic ℓ does not divide the orders of the components groups of the centralizers of the nilpotent elements. As an illustration, we rederive the modular Springer correspondence for general linear groups using this language.

From Section 4, we consider only classical groups defined over a field of odd characteristic. We recall results from [DJ92] defining a Dipper-James basic set datum for the Weyl group in the case $\ell \neq 2$, and the combinatorics of the ordinary Springer correspondence following [Lus84]. Then we want to compare the order considered by Dipper and James with the order induced by the Springer correspondence. However, it is easier to compare both of them to a third order, given by the dimensions of the Springer fibers, and which also has a combinatorial description. This implies that both methods lead to the same parametrization of the modular simple modules by a subset of the ordinary simple modules, and that one can deduce explicitly the modular Springer correspondence for $\ell \neq 2$ from the ordinary Springer correspondence.

In Section 5, we are actually able to compare directly the Dipper-James order with the Springer order.

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2. BASIC SET DATA

Let W be a finite group, ℓ a prime number, and $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ a sufficiently large ℓ -modular system, e.g. we can take for \mathbb{K} a finite extension of \mathbb{Q}_ℓ containing m -th roots of unity, where m is the exponent of W , with ring of integers \mathbb{O} and residue field \mathbb{F} . For $\mathbb{E} = \mathbb{K}$ or \mathbb{F} , we denote by $\mathbb{E}W$ the group algebra of W over \mathbb{E} , and by $\text{Irr } \mathbb{E}W$ a set of representatives of isomorphism classes of simple $\mathbb{E}W$ -modules. We have an ℓ -modular decomposition matrix

$$D^W := (d_{E,F}^W)_{E \in \text{Irr } \mathbb{K}W, F \in \text{Irr } \mathbb{F}W}$$

where $d_{E,F}^W$ is the composition multiplicity of the simple module F in $\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}$, where $E_{\mathbb{O}}$ is some integral form of E . This is independent of the choice of $E_{\mathbb{O}}$ [Ser67].

Definition 2.1. A basic set datum (for W) is a pair $\mathfrak{B} = (\leq, \beta)$ consisting of a partial order \leq on $\text{Irr } \mathbb{K}W$, and an injection $\beta : \text{Irr } \mathbb{F}W \hookrightarrow \text{Irr } \mathbb{K}W$ such that the following properties hold:

$$(2.1) \quad d_{\beta(F),F}^W = 1 \text{ for all } F \in \text{Irr } \mathbb{F}W;$$

$$(2.2) \quad d_{E,F}^W \neq 0 \Rightarrow E \leq \beta(F) \text{ for } E \in \text{Irr } \mathbb{K}W, F \in \text{Irr } \mathbb{F}W.$$

We will need to compare different basic set data. We adopt the convention that an order relation on a set X is a subset of the product $X \times X$. Thus we can consider the intersection of two orders, say that one order is included in another one, and so on.

Proposition 2.2. *Let us consider a fixed injection $\beta : \text{Irr } \mathbb{F}W \hookrightarrow \text{Irr } \mathbb{K}W$.*

- (1) *If (\leq_1, β) is a basic set datum and $\leq_1 \subseteq \leq_2$, then (\leq_2, β) is also a basic set datum.*
- (2) *If (\leq_1, β) and (\leq_2, β) are two basic set data, then $(\leq_1 \cap \leq_2, \beta)$ is also a basic set datum.*

Proof. This follows directly from the definitions. \square

Proposition 2.3. *Let $\mathfrak{B}_1 = (\leq_1, \beta_1)$ and $\mathfrak{B}_2 = (\leq_2, \beta_2)$ be two basic set data for W . Then, for any $F \in \text{Irr } \mathbb{F}W$, we have*

$$\beta_1(F) \leq_2 \beta_2(F) \leq_1 \beta_1(F).$$

If moreover $\leq_1 \subseteq \leq_2$, then $\beta_1 = \beta_2$.

Proof. Let F be an element of $\text{Irr } \mathbb{F}W$. By the property (2.1) for \mathfrak{B}_1 , we have $d_{\beta_1(F), F}^W = 1 \neq 0$, hence by the property (2.2) for \mathfrak{B}_2 , we have $\beta_1(F) \leq_2 \beta_2(F)$. Similarly, by the property (2.1) for \mathfrak{B}_2 , we have $d_{\beta_2(F), F}^W = 1 \neq 0$, hence by the property (2.2) for \mathfrak{B}_1 , we have $\beta_2(F) \leq_1 \beta_1(F)$. Thus $\beta_1(F) \leq_2 \beta_2(F) \leq_1 \beta_1(F)$.

If moreover $\leq_1 \subseteq \leq_2$, then $\beta_1(F) \leq_2 \beta_2(F) \leq_2 \beta_1(F)$, hence $\beta_1(F) = \beta_2(F)$, for any F . \square

Corollary 2.4. *Given a partial order \leq on $\text{Irr } \mathbb{K}W$, there is at most one injection $\beta : \text{Irr } \mathbb{F}W \hookrightarrow \text{Irr } \mathbb{K}W$ such that (\leq, β) is a basic set datum.*

3. SPRINGER BASIC SETS

3.1. Modular Springer correspondence. Let us consider a reductive group G over $\overline{\mathbb{F}}_p$, endowed with an \mathbb{F}_q -rational structure given by a Frobenius endomorphism F , where q is some power of the prime p . It acts by the adjoint action on its Lie algebra \mathfrak{g} , and there are finitely many orbits in the nilpotent cone $\mathcal{N} \subseteq \mathfrak{g}$. For x in \mathcal{N} , we denote by \mathcal{O}_x the G -orbit of x . The set of nilpotent orbits is partially ordered by the following relation: $\mathcal{O}_1 \leq \mathcal{O}_2$ if and only if $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$. For each orbit $\mathcal{O} \subseteq \mathcal{N}$, we choose a representative $x_{\mathcal{O}}$, and we denote by $A_G(\mathcal{O})$ the group of components of the centralizer $C_G(x_{\mathcal{O}})$, that is, $A_G(\mathcal{O}) := C_G(x_{\mathcal{O}})/C_G^0(x_{\mathcal{O}})$.

As in the last section, we choose an ℓ -modular system $(\mathbb{K}, \mathbb{O}, \mathbb{F})$, and we assume $\ell \neq p$. For $\mathbb{E} = \mathbb{K}$ or \mathbb{F} , let $\mathfrak{P}_{\mathbb{E}}^G$, or $\mathfrak{P}_{\mathbb{E}}$ if G is clear from the context, denote the set of all pairs $(\mathcal{O}, \mathcal{L})$ where \mathcal{O} is a nilpotent orbit in \mathfrak{g} and \mathcal{L} is an irreducible G -equivariant \mathbb{E} -local system on it. We will identify $\mathfrak{P}_{\mathbb{E}}$ with the set of pairs $(x_{\mathcal{O}}, \rho)$ where $\rho \in \text{Irr } \mathbb{E}A_G(x_{\mathcal{O}})$. The simple G -equivariant perverse sheaves on \mathcal{N} with \mathbb{E} -coefficients are intersection cohomology complexes parametrized by $\mathfrak{P}_{\mathbb{E}}$. We denote them by $\mathbf{IC}(\overline{\mathcal{O}}, \mathcal{L})$, or $\mathbf{IC}(x_{\mathcal{O}}, \rho)$. Similarly to the case of a finite group, one can define a decomposition matrix [Jut09]

$$D^{\mathcal{N}} := (d_{(x,\rho),(y,\sigma)}^{\mathcal{N}})_{(x,\rho) \in \mathfrak{P}_{\mathbb{K}}, (y,\sigma) \in \mathfrak{P}_{\mathbb{F}}}$$

for G -equivariant perverse sheaves on \mathcal{N} , where $d_{(x,\rho),(y,\sigma)}^{\mathcal{N}}$ is the composition multiplicity of $\mathbf{IC}(y, \sigma)$ in $\mathbb{F} \otimes_{\mathbb{O}}^L \mathbf{IC}(x, \rho_{\mathbb{O}})$ for any choice of an integral form $\rho_{\mathbb{O}}$ of the local system ρ .

Let W be the Weyl group of G . From now on, we assume that p is very good for G [Car93, p.29] (thus for classical groups of type BCD , we only have to assume $p \neq 2$). If $\mathbb{E} = \mathbb{K}$ or \mathbb{F} , one can naturally define an injective map

$$(3.1) \quad \Psi_{\mathbb{E}} : \text{Irr } \mathbb{E}W \hookrightarrow \mathfrak{P}_{\mathbb{E}}$$

using a Fourier-Deligne transform. In the ordinary case, it is the Springer correspondence of [Bry86] (and coincides with the one of [Spr76]), and in the modular case it was first introduced in [Jut07].

Remark 3.1. The injectivity of $\Psi_{\mathbb{E}}$ comes from the fact that there is an inverse Fourier transform.

Remark 3.2. We have $\Psi_{\mathbb{E}}(1_W) = (0, 1)$ because the Fourier-Deligne transform of the constant sheaf is a sky-scraper sheaf supported at the origin, just as the Fourier transform of a constant function is a Dirac distribution supported at the origin.

Remark 3.3. What is usually called Springer correspondence (but only in the case $\mathbb{E} = \mathbb{K}$) is the one defined in [Lus81, BM81], and for which one can find tables in [Car93]. We will denote this map by

$$\Phi_{\mathbb{K}} : \text{Irr } \mathbb{K}W \hookrightarrow \mathfrak{P}_{\mathbb{K}}.$$

Note that this construction makes sense both for the group G and for its Lie algebra \mathfrak{g} . Since the characteristic p is very good for G , we can also use a Springer isomorphism from the unipotent variety to the nilpotent cone to transfer the construction for the group to the Lie algebra, but by [Sho88, 5.3] this gives the same result as working directly with the Lie algebra.

The maps $\Phi_{\mathbb{K}}$ and $\Psi_{\mathbb{K}}$ differ by the sign character ε of the Weyl group:

Proposition 3.4. [Sho88, 17.7] *For any $E \in \text{Irr } \mathbb{K}W$, we have*

$$\Psi_{\mathbb{K}}(E) = \Phi_{\mathbb{K}}(\varepsilon \otimes E).$$

The following theorem from [Jut07] shows that the ordinary and modular Springer correspondences respect decomposition numbers.

Theorem 3.5. *For $E \in \text{Irr } \mathbb{K}W$ and $F \in \text{Irr } \mathbb{F}W$, we have*

$$d_{E,F}^W = d_{\Psi_{\mathbb{K}}(E), \Psi_{\mathbb{F}}(F)}^{\mathcal{N}}.$$

Hence D^W can be seen as a submatrix of $D^{\mathcal{N}}$. Now $D^{\mathcal{N}}$ has the following easy property, considering supports [Jut, Corollary 2.6 and Proposition 2.7].

Proposition 3.6. *For $(x, \rho) \in \mathfrak{P}_{\mathbb{K}}$ and $(y, \sigma) \in \mathfrak{P}_{\mathbb{F}}$, we have*

$$(3.2) \quad d_{(x,\rho),(y,\sigma)}^{\mathcal{N}} = 0 \text{ unless } \mathcal{O}_y \leq \mathcal{O}_x,$$

$$(3.3) \quad d_{(x,\rho),(y,\sigma)}^{\mathcal{N}} = d_{\rho,\sigma}^{A_G(x)} \text{ if } y = x.$$

In the preceding formula, we have denoted by $d_{\rho,\sigma}^{A_G(x)}$ the decomposition numbers for the finite group $A_G(x)$. In order to define basic sets using the Springer correspondence, the following result plays a crucial role [Jut, Proposition 5.4].

Proposition 3.7. *If $(y, \sigma) \in \text{Im } \Psi_{\mathbb{F}}$ and $d_{(x,\rho),(y,\sigma)}^{\mathcal{N}} \neq 0$, then $(x, \rho) \in \text{Im } \Psi_{\mathbb{K}}$.*

3.2. Definition of Springer basic sets. Suppose we can choose a family $\mathfrak{B}_{\bullet} = (\mathfrak{B}_{\mathcal{O}})$ of basic set data for the groups $A_G(\mathcal{O})$. For each nilpotent orbit \mathcal{O} , we write $\mathfrak{B}_{\mathcal{O}} = (\leq_{\mathcal{O}}, \beta_{\mathcal{O}})$. To such a family \mathfrak{B}_{\bullet} , we will associate a basic set datum $(\leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}, \beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}})$ for W , the *Springer basic set datum*.

Remark 3.8. If G is simple and of adjoint type (which we may assume since we study Springer correspondence in a modular, but not generalized setting, in the sense of Lusztig), then the $A_G(\mathcal{O})$ are as follows: trivial in type A_n , elementary abelian 2-groups in other classical types, and symmetric groups of rank at most 5 in exceptional types. We know that these groups admit basic sets. In the sequel it will be combinatorially more convenient to work with special orthogonal and symplectic groups (rather than the adjoint groups), but for those the $A_G(\mathcal{O})$ are also elementary abelian 2-groups.

Remark 3.9. In the case where ℓ does not divide the orders of the groups $A_G(x)$, we will see that much of the discussion simplifies (see Paragraph 3.3). In particular, there will be no need to choose basic set data for each orbit: in that case, we may assume that we take the trivial order everywhere, and the $\beta_{\mathcal{O}}$ are canonical bijections (in that case, the Brauer characters are just the ordinary characters).

Definition 3.10. We define the partial order $\leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}$ on $\text{Irr } \mathbb{K}W$ by the following rule. For $i \in \{1, 2\}$, let $E_i \in \text{Irr } \mathbb{K}W$, and let us write $\Psi_{\mathbb{K}}(E_i) = (\mathcal{O}_i, \rho_i)$. Then

$$E_1 \leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}} E_2 \iff \begin{cases} \text{either } \mathcal{O}_2 < \mathcal{O}_1, \\ \text{or } (\mathcal{O}_1 = \mathcal{O}_2 =: \mathcal{O} \text{ and } \rho_1 \leq_{\mathcal{O}} \rho_2) \end{cases}$$

It is clear that this defines indeed an order relation on $\text{Irr } \mathbb{K}W$ (for the antisymmetry, one needs to use the injectivity of $\Psi_{\mathbb{K}}$).

Proposition 3.11. *Let $F \in \text{Irr } \mathbb{F}W$, and let us write $\Psi_{\mathbb{F}}(F) = (\mathcal{O}, \sigma)$. Then there exists a unique $E \in \text{Irr } \mathbb{K}W$ such that $\Psi_{\mathbb{K}}(E) = (\mathcal{O}, \beta_{\mathcal{O}}(\sigma))$.*

Proof. We have $(\mathcal{O}, \sigma) = \Psi_{\mathbb{F}}(F) \in \text{Im } \Psi_{\mathbb{F}}$, and

$$d_{(\mathcal{O}, \beta_{\mathcal{O}}(\sigma)), (\mathcal{O}, \sigma)}^{\mathcal{N}} = d_{\beta_{\mathcal{O}}(\sigma), \sigma}^{A_G(\mathcal{O})} = 1 \neq 0,$$

hence $(\mathcal{O}, \beta_{\mathcal{O}}(\sigma)) \in \text{Im } \Psi_{\mathbb{K}}$ by Proposition 3.7. This proves the existence of E . The uniqueness of E follows from the injectivity of $\Psi_{\mathbb{K}}$. \square

Definition 3.12. We define the map $\beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}} : \text{Irr } \mathbb{F}W \rightarrow \text{Irr } \mathbb{K}W$ by associating to each $F \in \text{Irr } \mathbb{F}W$ the module $E \in \text{Irr } \mathbb{K}W$ provided by Proposition 3.11.

In other words, $\beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}$ completes the following commutative diagram :

$$\begin{array}{ccccc} \text{Irr } \mathbb{F}W & \xrightarrow{\Psi_{\mathbb{F}}} & \mathfrak{P}_{\mathbb{F}} & \xrightarrow{\sim} & \sqcup_{\mathcal{O}} \text{Irr } \mathbb{F}A_G(\mathcal{O}) \\ \beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}} \downarrow & & \downarrow \beta_{\bullet} & & \downarrow \sqcup_{\mathcal{O}} \beta_{\mathcal{O}} \\ \text{Irr } \mathbb{K}W & \xrightarrow{\Psi_{\mathbb{K}}} & \mathfrak{P}_{\mathbb{K}} & \xrightarrow{\sim} & \sqcup_{\mathcal{O}} \text{Irr } \mathbb{K}A_G(\mathcal{O}) \end{array}$$

where β_{\bullet} is defined by the commutativity of the right square. The $\beta_{\mathcal{O}}$ are injective, thus β_{\bullet} is injective, and the composition $\beta_{\bullet} \circ \Psi_{\mathbb{F}} = \Psi_{\mathbb{K}} \circ \beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}$ is injective as well. It follows that $\beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}$ is injective.

Theorem 3.13. *Assume we have chosen a family $\mathfrak{B}_{\bullet} = (\mathfrak{B}_{\mathcal{O}})$ of basic set data for the groups $A_G(\mathcal{O})$, where \mathcal{O} runs over all nilpotent orbits. Then the pair $(\leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}, \beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}})$ defined above is a basic set datum for W .*

Proof. Let $F \in \text{Irr } \mathbb{F}W$. Let us write $(\mathcal{O}, \sigma) := \Psi_{\mathbb{F}}(F)$, and let $E_F := \beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}(F)$. By definition, we have $\Psi_{\mathbb{K}}(E_F) = (\mathcal{O}, \beta_{\mathcal{O}}(\sigma))$. Thus

$$d_{\beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}(F), F}^W = d_{E_F, F}^W = d_{(\mathcal{O}, \beta_{\mathcal{O}}(\sigma)), (\mathcal{O}, \sigma)}^{\mathcal{N}} = d_{\beta_{\mathcal{O}}(\sigma), \sigma}^{A_G(\mathcal{O})} = 1.$$

Now, in addition, let $E \in \mathbb{K}W$, and assume $d_{E, F}^W \neq 0$. Let us write $(\mathcal{O}_E, \rho_E) := \Psi_{\mathbb{K}}(E)$. By Theorem 3.5, we have

$$d_{(\mathcal{O}_E, \rho_E), (\mathcal{O}, \sigma)}^{\mathcal{N}} = d_{E, F}^W \neq 0$$

hence $\mathcal{O}_E \geq \mathcal{O}$ by Proposition 3.6. Now, if $\mathcal{O}_E = \mathcal{O}$, then we have

$$0 \neq d_{(\mathcal{O}, \rho_E), (\mathcal{O}, \sigma)}^{\mathcal{N}} = d_{\rho_E, \sigma}^{A_G(\mathcal{O})}$$

again by Proposition 3.6, hence $\rho_E \leq_{\mathcal{O}} \beta_{\mathcal{O}}(\sigma)$ since $\mathfrak{B}_{\mathcal{O}} = (\leq_{\mathcal{O}}, \beta_{\mathcal{O}})$ is a basic set for $A_G(\mathcal{O})$. This proves that $E \leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}} E_F = \beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}(F)$.

Thus $(\leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}, \beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}})$ is a basic set datum for W . \square

The pair $(\leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}, \beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}})$ is the Springer basic set datum for W associated to the choice of the family \mathfrak{B}_{\bullet} , and we call the image of $\beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}$ a Springer basic set.

3.3. Good case. In this paragraph, we assume that ℓ does not divide the orders of the groups $A_G(\mathcal{O})$, where \mathcal{O} runs over the nilpotent orbits in \mathfrak{g} . Then the algebras $\mathbb{F}A_G(\mathcal{O})$ are semisimple and the decomposition matrices $D^{A_G(\mathcal{O})}$ are equal to the identity. Thus we can take the trivial order (that is, the identity relation) $=_{\mathcal{O}}$ on $\text{Irr } \mathbb{K}A_G(\mathcal{O})$, and $\beta_{\mathcal{O}}$ is a canonical bijection. By abuse of notation, we will identify the sets $\mathfrak{P}_{\mathbb{K}}$ and $\mathfrak{P}_{\mathbb{F}}$ with a common index set \mathfrak{P} . Let us reformulate the definition of Springer basic sets in this favorable situation.

The order $\leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}$, in this case, is the order $\leq_{\text{triv}}^{\mathcal{N}}$ defined by

$$E_1 \leq_{\text{triv}}^{\mathcal{N}} E_2 \iff (E_1 = E_2 \text{ or } \mathcal{O}_2 < \mathcal{O}_1)$$

where $E_i \in \text{Irr } \mathbb{K}W$ and $(\mathcal{O}_i, \mathcal{L}_i) := \Psi_{\mathbb{K}}(E_i)$, $i = 1, 2$.

The injection $\beta_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}$ is the map $\beta_{\ell} : \text{Irr } \mathbb{F}W \hookrightarrow \text{Irr } \mathbb{K}W$ defined in the following way: to $F \in \text{Irr } \mathbb{F}W$, we associate the unique $E := \beta_{\ell}(F) \in \text{Irr } \mathbb{K}W$ such that $\Psi_{\mathbb{K}}(E) = \Psi_{\mathbb{F}}(F) \in \mathfrak{P}$.

Then the pair $\mathfrak{B}_{\ell} := (\leq_{\text{triv}}^{\mathcal{N}}, \beta_{\ell})$ is a basic set datum for W . Moreover, any choice of a family \mathfrak{B}_{\bullet} of basic set data for the $A_G(\mathcal{O})$ will give rise to a basic set datum for W with the same injection β_{ℓ} , since the trivial order $=_{\mathcal{O}}$ is contained in any chosen order $\leq_{\mathcal{O}}$, and thus the order $\leq_{\text{triv}}^{\mathcal{N}}$ is contained in $\leq_{\mathfrak{B}_{\bullet}}^{\mathcal{N}}$, and Proposition 2.3 applies.

Remark 3.14. Basic sets for Weyl groups (and actually for Hecke algebras) in the good prime case have already been constructed, for example in [GR01]. For symmetric groups, this goes back to James [Jam76].

3.4. The case of the general linear group. Although the modular Springer correspondence for \mathbf{GL}_n has been determined in [Jut07], we will do it again here in the language of Springer basic sets, as this case is much simpler than the other classical types, for which it will serve as a model.

Let \mathfrak{P}_n denote the set of all partitions of n . The prime ℓ being fixed, the subset $\mathfrak{P}_n^{(\ell)}$ consists of ℓ -regular partitions, that is, those which do not contain entries repeated at least ℓ times. The notation $\lambda \vdash n$, resp. $\lambda \vdash^{(\ell)} n$, means $\lambda \in \mathfrak{P}_n$, resp. $\lambda \in \mathfrak{P}_n^{(\ell)}$.

Nilpotent orbits for \mathbf{GL}_n are parametrized by \mathfrak{P}_n via the Jordan normal form, and the orbit closure order is given by the classical dominance order on partitions.

It turns out that the simple modules of $\mathbb{K}\mathfrak{S}_n$, called Specht modules, are also parametrized by \mathfrak{P}_n . In [Jam76], James classifies the simple modules for $\mathbb{F}\mathfrak{S}_n$. Let us say a few words about this. The Specht module S^λ is defined as a submodule of the permutation module M^λ of \mathfrak{S}_n on the parabolic subgroup \mathfrak{S}_λ corresponding to the partition λ . The permutation module M^λ is endowed with a natural scalar product, which restricts to a scalar product on S^λ . The module and the bilinear form are defined over \mathbb{O} (actually over \mathbb{Z}), and thus one can reduce them to get a module for $\mathbb{F}\mathfrak{S}_n$, still called a Specht module, endowed with a bilinear form, which no longer needs to be non-degenerate. Then the quotient of the Specht module S^λ by the radical of the bilinear form is either zero or a simple module denoted by D^λ . The partitions giving a non-zero result are exactly the ℓ -regular partitions. The D^λ , for $\lambda \in \mathfrak{P}_n^{(\ell)}$, form a complete set of representatives of $\mathbb{F}\mathfrak{S}_n$ -modules. Moreover, James shows that (in our language) the pair consisting of the dominance order on partitions, and the injection sending D^λ ($\lambda \in \mathfrak{P}_n^{(\ell)}$) to S^λ , is a basic set datum, which we will call the James basic set datum.

Now, the ordinary Springer correspondence sends the Specht module S^λ to the orbit \mathcal{O}_{λ^*} (with the trivial local system), where λ^* denotes the conjugate partition, and the orbit closure order is again the dominance order. The Springer and James basic set data involve the same order relation, hence they coincide by Corollary 2.4. We have deduced the modular Springer correspondence for \mathbf{GL}_n .

Theorem 3.15. *The modular Springer correspondence for \mathbf{GL}_n sends the simple module D^μ ($\mu \in \mathfrak{P}_n^{(\ell)}$) to the orbit \mathcal{O}_{μ^*} (with the trivial local system).*

To simplify the notation, we set $d_{\lambda,\mu}^{\mathfrak{S}_n} := d_{S^\lambda, D^\mu}^{\mathfrak{S}_n}$ and $d_{\lambda,\mu}^{\mathcal{N}} := d_{(\mathcal{O}_\lambda, 1), (\mathcal{O}_\mu, 1)}^{\mathcal{N}}$. Theorems 3.5 and 3.15 imply that

$$d_{\lambda,\mu}^{\mathfrak{S}_n} = d_{\lambda^*, \mu^*}^{\mathcal{N}}$$

for $\lambda \vdash n$ and $\mu \vdash^{(\ell)} n$.

4. MODULAR SPRINGER CORRESPONDENCE FOR CLASSICAL TYPES

From now on, we assume $\ell \neq 2$. Under that hypothesis, we will see in this section that, for a group of classical type, the Springer basic sets coincide with those given by the classical theory of Dipper and James, leading to a parametrization of simple $\mathbb{F}W$ -modules by pairs of ℓ -regular partitions. This determines simply the modular Springer correspondence in this case. Note that situation is different for $\ell = 2$.

4.1. Dipper-James orders on bipartitions.

4.1.1. *Type B_n/C_n .* Let W_n be the Weyl group of type B_n . We can identify it with the group of signed permutations of the set $\{\pm 1, \dots, \pm n\}$. The classification of ordinary and modular simple modules is completely analogous to the case of the symmetric group [DJ92]. We have a parametrization $\text{Irr } \mathbb{K}W_n = \{\mathbf{S}^\lambda \mid \lambda \in \mathfrak{Bip}_n\}$, where \mathfrak{Bip}_n denotes the set of all bipartitions of n , that is, the set of pairs of partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| = n$. For example, the trivial representation is labeled by the pair $((n), \emptyset)$ and the sign representation by $((\emptyset), (1^n))$.

Again, those modules are defined over \mathbb{Z} and endowed with a bilinear form, and factoring out the radical of the form yields 0 or a simple module, and one obtains a complete collection of simple $\mathbb{F}W_n$ -modules in this way. In our case ($\ell \neq 2$), the result is that we have a parametrization $\text{Irr } \mathbb{F}W_n = \{\mathbf{D}^\lambda \mid \lambda \in \mathfrak{Bip}_n^{(\ell)}\}$, where $\mathfrak{Bip}_n^{(\ell)}$ denotes the set of ℓ -regular bipartitions of n , that is, the bipartitions $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ such that $\lambda^{(1)}$ and $\lambda^{(2)}$ are ℓ -regular. Naming the modular simple modules in this way implicitly amounts to define an injection γ_ℓ from $\text{Irr } \mathbb{F}W_n$ to $\text{Irr } \mathbb{K}W_n$, sending \mathbf{D}^λ to \mathbf{S}^λ , for $\lambda \in \mathfrak{Bip}_n^{(\ell)}$.

By [DJ92], a Morita equivalence allows to express the decomposition numbers $d_{\lambda,\mu}^{W_n} := d_{\mathbf{S}^\lambda, \mathbf{D}^\mu}^{W_n}$ of W_n in terms of decomposition numbers for the symmetric groups \mathfrak{S}_r for $0 \leq r \leq n$.

Proposition 4.1. [DJ92] *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ and $\mu = (\mu^{(1)}, \mu^{(2)})$ be two bipartitions of n , with μ ℓ -regular. If $|\lambda^{(1)}| = |\mu^{(1)}| =: r$, then*

$$d_{\lambda,\mu}^{W_n} = d_{\lambda^{(1)}, \mu^{(1)}}^{\mathfrak{S}_r} \cdot d_{\lambda^{(2)}, \mu^{(2)}}^{\mathfrak{S}_{n-r}}$$

and otherwise we have $d_{\lambda,\mu}^{W_n} = 0$.

Definition 4.2 (Dipper-James order on bipartitions). Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ and $\mu = (\mu^{(1)}, \mu^{(2)})$ be two bipartitions of n . We say that $\lambda \leq_{DJ} \mu$ if, for $i \in \{1, 2\}$, we have $|\lambda^{(i)}| = |\mu^{(i)}|$ and $\lambda^{(i)} \leq \mu^{(i)}$. This induces a partial order on $\text{Irr } \mathbb{K}W_n$ that we still denote by \leq_{DJ} .

By the results for the symmetric group, the preceding proposition implies:

Proposition 4.3. *The pair (\leq_{DJ}, γ_ℓ) is a basic set datum for W_n .*

We will call it the Dipper-James basic set datum.

Since the two versions of the ordinary Springer correspondence are related by tensoring with the sign character ε , it is useful to recall how this translates combinatorially. If $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is a bipartition, we set $\lambda^* := (\lambda^{(2)*}, \lambda^{(1)*})$.

Proposition 4.4 ([GP00], Theorem 5.5.6). *For λ in \mathfrak{Bip}_n , we have $\varepsilon \otimes \mathbf{S}^\lambda = \mathbf{S}^{\lambda^*}$.*

Remark 4.5. We have $\lambda \leq_{DJ} \mu \Leftrightarrow \mu^* \leq_{DJ} \lambda^*$.

4.1.2. *Type D_n .* For $n \geq 2$, let W'_n be the index two subgroup of W_n consisting of signed permutations with an even number of sign flips. It is a Weyl group of type D_n .

We have seen in 4.1.1 that $\text{Irr } \mathbb{K}W_n$ is parametrized by \mathfrak{Bip}_n . As in 4.1.1, we denote by $\mathbf{S}^{(\lambda^{(1)}, \lambda^{(2)})}$ the simple $\mathbb{K}W_n$ -module labeled by $(\lambda^{(1)}, \lambda^{(2)})$. A classification of the simple $\mathbb{K}W'_n$ -modules is obtained as follows:

- If $\lambda^{(1)} \neq \lambda^{(2)}$, then $\mathbf{S}^{(\lambda^{(1)}, \lambda^{(2)})}$ and $\mathbf{S}^{(\lambda^{(2)}, \lambda^{(1)})}$ have the same restriction to W'_n and this restriction is a simple $\mathbb{K}W'_n$ -module which we will denote by $\mathbf{S}^{[\lambda^{(1)}, \lambda^{(2)}]}$.
- The restriction of $\mathbf{S}^{(\lambda, \lambda)}$ splits into a direct sum of two non-isomorphic simple $\mathbb{K}W'_n$ -modules which are denoted by $\mathbf{S}^{[\lambda, +]}$ and $\mathbf{S}^{[\lambda, -]}$ (this case arises only for even n).

Moreover, every simple $\mathbb{K}W'_n$ -module arises exactly once in this way. Note that we use the notation (a, b) for an ordered pair, and $[a, b]$ for an unordered pair (that is, an equivalence class of pairs with respect to swapping a and b). Let $\overline{\mathfrak{Bip}}_n$ denote the set of unordered bipartitions of n , and let

$$\widetilde{\mathfrak{Bip}}_n = \begin{cases} \overline{\mathfrak{Bip}}_n & \text{if } n \text{ is odd} \\ \{[\lambda_1, \lambda_2] \in \overline{\mathfrak{Bip}}_n \mid \lambda_1 \neq \lambda_2\} \cup \{[\lambda, \pm] \mid \lambda \in \mathfrak{P}_{\frac{n}{2}}\} & \text{if } n \text{ is even.} \end{cases}$$

By the above, $\text{Irr } \mathbb{K}W'_n$ can be parametrized by $\widetilde{\mathfrak{Bip}}_n$. We denote by $\overline{\mathfrak{Bip}}_n^{(\ell)}$ the set of ℓ -regular unordered bipartitions (those where each partition is ℓ -regular), and similarly for $\widetilde{\mathfrak{Bip}}_n^{(\ell)}$.

Recall from 4.1.1 that $\mathbf{D}^{(\lambda^{(1)}, \lambda^{(2)})}$ denotes the simple $\mathbb{F}W_n$ -module parametrized by $(\lambda^{(1)}, \lambda^{(2)}) \in \mathfrak{Bip}_n^{(\ell)}$.

Definition 4.6. For any $E \in \text{Irr } \mathbb{K}W'_n$ we denote by a_E the a -invariant of E (cf [GP00] definitions 6.5.7 and 9.4.8). For any $F \in \text{Irr } \mathbb{F}W'_n$, we define

$$a_F := \min\{a_E \mid E \in \text{Irr } \mathbb{K}W'_n \text{ and } d_{E,F}^{W'_n} \neq 0\}.$$

Proposition 4.7. (*Relation between $\text{Irr } \mathbb{F}W_n$ and $\text{Irr } \mathbb{F}W'_n$ [Gec00].*)

(1) *There is a parametrization*

$$\text{Irr } \mathbb{F}W'_n = \{\mathbf{E}^\lambda \mid \lambda \in \widetilde{\mathfrak{Bip}}_n^{(\ell)}\}$$

such that, if $\gamma'_\ell : \text{Irr } \mathbb{F}W'_n \rightarrow \text{Irr } \mathbb{K}W'_n$ is induced by the natural inclusion $\widetilde{\mathfrak{Bip}}_n^{(\ell)} \rightarrow \mathfrak{Bip}_n$, the following properties hold:

- *For all $F \in \text{Irr } \mathbb{F}W'_n$, we have $d_{\gamma'_\ell(F), F}^{W'_n} = 1$ and $a_{\gamma'_\ell(F)} = a_F$,*
- *Given $E \in \text{Irr } \mathbb{K}W'_n$ and $F \in \text{Irr } \mathbb{F}W'_n$, we have*

$$d_{E,F}^{W'_n} \neq 0 \Rightarrow a_F < a_E \text{ or } E = \gamma'_\ell(F).$$

(2) *Moreover, we have the following relations between the simple modules for $\mathbb{F}W_n$ and for $\mathbb{F}W'_n$.*

- *If $\lambda^{(1)} \neq \lambda^{(2)}$, then $\mathbf{D}^{(\lambda^{(1)}, \lambda^{(2)})}$ and $\mathbf{D}^{(\lambda^{(2)}, \lambda^{(1)})}$ have the same restriction to W'_n and this restriction is the simple $\mathbb{F}W'_n$ -module $\mathbf{E}^{[\lambda^{(1)}, \lambda^{(2)}]}$,*
- *the restriction of $\mathbf{D}^{(\lambda, \lambda)}$ splits into a direct sum of two non-isomorphic simple $\mathbb{F}W'_n$ -modules which are $\mathbf{E}^{[\lambda, +]}$ and $\mathbf{E}^{[\lambda, -]}$.*

Definition 4.8 (Dipper-James order on unordered bipartitions). We define a partial order on the set $\overline{\mathfrak{Bip}}_n$ of unordered bipartitions of n by:

$$\begin{aligned} [\lambda^{(1)}, \lambda^{(2)}] &\leq_{DJ} [\mu^{(1)}, \mu^{(2)}] \\ &\quad \Updownarrow \\ (\lambda^{(1)}, \lambda^{(2)}) &\leq_{DJ} (\mu^{(1)}, \mu^{(2)}) \text{ or } (\lambda^{(1)}, \lambda^{(2)}) \leq_{DJ} (\mu^{(2)}, \mu^{(1)}) \end{aligned}$$

(It is easy to check that this is indeed a partial order.)

We have a natural projection $\varphi : \widetilde{\mathfrak{Bip}}_n \rightarrow \overline{\mathfrak{Bip}}_n$, sending $[\lambda, \pm]$ to $[\lambda, \lambda]$. The order \leq_{DJ} on $\overline{\mathfrak{Bip}}_n$ induces an order that we still denote by \leq_{DJ} on $\widetilde{\mathfrak{Bip}}_n$, hence on $\text{Irr } \mathbb{K}W'_n$, such that $\lambda < \mu$ if and only if

$\varphi(\boldsymbol{\lambda}) < \varphi(\boldsymbol{\mu})$ for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \widetilde{\mathfrak{Bip}}_n$. So the irreducible modules $\mathbf{S}^{[\lambda, +]}$ and $\mathbf{S}^{[\lambda, -]}$ are not comparable.

Proposition 4.9. *The pair $(\leq_{DJ}, \gamma'_\ell)$ is a basic set datum for W'_n .*

To prove the proposition, we will need the fact that the restriction from W_n to W'_n commutes with decomposition maps. Let us introduce some notation. If A is a finite dimensional algebra over a field, we denote by $R_0(A)$ the Grothendieck group of the category of finite dimensional A -modules. The class of an A -module V in $R_0(A)$ is denoted by $[V]$.

Lemma 4.10 ([Gec00], lemma 5.2). *The restriction of modules from $\text{Irr } \mathbb{K}W_n$ to $\text{Irr } \mathbb{K}W'_n$ (resp. from $\text{Irr } \mathbb{F}W_n$ to $\text{Irr } \mathbb{F}W'_n$) induces maps between Grothendieck groups fitting into the following commutative diagram*

$$\begin{array}{ccc} R_0(\mathbb{K}W_n) & \xrightarrow{\text{Res}} & R_0(\mathbb{K}W'_n) \\ \downarrow d & & \downarrow d' \\ R_0(\mathbb{F}W_n) & \xrightarrow{\text{Res}} & R_0(\mathbb{F}W'_n) \end{array}$$

where d and d' denote the decomposition maps.

Proof of Proposition 4.9. By the lemma, for $\lambda^{(1)} \neq \lambda^{(2)}$, we have

$$\begin{aligned} d'([[\mathbf{S}^{[\lambda^{(1)}, \lambda^{(2)}]]}) &= \text{Res}(d([\mathbf{S}^{(\lambda^{(1)}, \lambda^{(2)})}])) \\ &= \text{Res}([\mathbf{D}^{(\lambda^{(1)}, \lambda^{(2)})}]) + \text{Res}(\text{upper terms for } \leq_{DJ}) \\ &= [\mathbf{E}^{[\lambda^{(1)}, \lambda^{(2)}]}] + (\text{upper terms for } \leq_{DJ}) \end{aligned}$$

For $\lambda^{(1)} = \lambda^{(2)} = \lambda$, we have

$$d'([\mathbf{S}^{[\lambda, +]}] + [\mathbf{S}^{[\lambda, -]}]) = [\mathbf{E}^{[\lambda, +]}] + [\mathbf{E}^{[\lambda, -]}] + \text{upper terms for } \leq_{DJ}$$

By Proposition 4.7 (1), we get $d_{\mathbf{S}^{[\lambda, +]}, \mathbf{E}^{[\lambda, -]}}^{W'_n} = 0$ as $\mathbf{S}^{[\lambda, -]}$ and $\mathbf{S}^{[\lambda, +]}$ have the same a -invariant.

This proves that $(\leq_{DJ}, \gamma'_\ell)$ is a basic set datum. \square

Proposition 4.11 ([GP00], Remark 5.6.5). *Let $\boldsymbol{\lambda} = [\lambda^{(1)}, \lambda^{(2)}]$ be in $\widetilde{\mathfrak{Bip}}_n$ such that $\lambda^{(1)} \neq \lambda^{(2)}$, then $\varepsilon \otimes \mathbf{S}^\lambda = \mathbf{S}^{\lambda^*}$ where $\lambda^* := [\lambda^{(1)*}, \lambda^{(2)*}]$.*

Let $\boldsymbol{\lambda} = [\lambda, \lambda]$ be in $\widetilde{\mathfrak{Bip}}_n$, then $\varepsilon \otimes \mathbf{S}^{[\lambda, \pm]} = \mathbf{S}^{[\lambda^, \pm]}$.*

Remark 4.12. Let $\boldsymbol{\lambda}, \boldsymbol{\mu}$ be in $\widetilde{\mathfrak{Bip}}_n$, then, $\boldsymbol{\lambda} \leq_{DJ} \boldsymbol{\mu} \Leftrightarrow \boldsymbol{\mu}^* \leq_{DJ} \boldsymbol{\lambda}^*$.

4.2. Nilpotent orbits and Lusztig's symbols. We now recall the definition of Lusztig's symbols [Lus84].

4.2.1. *Lusztig's symbols in type C.* Let N be an even integer ≥ 1 . We will introduce symbols which parametrize \mathfrak{P}^G for $G = \mathbf{Sp}_N$. They are partitioned according to their *defect* d , an odd integer ≥ 1 . This gives the partition into the different series occurring for the generalized Springer correspondence. The non-generalized correspondence that we study in this article will be the $d = 1$ case.

Let \mathbf{Symb}_N^d be the set of all ordered pairs $\binom{S}{T}$, where S is a finite subset of $\{0, 1, 2, \dots\}$ and T is a finite subset of $\{1, 2, 3, \dots\}$ such that:

- $l(S) = l(T) + d$ (where $l(S)$ is the number of elements in S),
- S and T contain no consecutive integers,
- $\sum_{s \in S} s + \sum_{t \in T} t = \frac{N}{2} + \frac{1}{2}(l(S) + l(T))(l(S) + l(T) - 1)$.

Moreover, these pairs are taken modulo the equivalence relation generated by the shift operation:

$$\binom{S}{T} \simeq \binom{\{0\} \cup (S + 2)}{\{1\} \cup (T + 2)}$$

We set $\mathbf{Symb}_N = \bigcup_{d \geq 1 \text{ odd}} \mathbf{Symb}_N^d$. We now define the δ function on

symbols, which will correspond to the dimension of the corresponding Springer fiber. Given a symbol Λ in \mathbf{Symb}_N , write the entries of Λ in a single row in increasing order:

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq x_{2m}$$

Let $x_0^0 \leq x_1^0 \leq x_2^0 \leq \dots \leq x_{2m}^0$ be the sequence

$$0 \leq 1 \leq 2 \leq 3 \leq \dots \leq 2m - 2 \leq 2m - 1 \leq 2m.$$

We set

$$\delta(\Lambda) := \sum_{0 \leq i < j \leq 2m} \inf(x_i, x_j) - \sum_{0 \leq i < j \leq 2m} \inf(x_i^0, x_j^0).$$

By an easy computation, we get:

$$\delta(\Lambda) = \sum_{0 \leq i < j \leq 2m} \inf(x_i, x_j) - \frac{1}{3}m(4m^2 - 1).$$

4.2.2. *Lusztig's symbols in type B and D.* We now introduce symbols which will parametrize \mathfrak{P}^G for $G = \mathbf{SO}_N$. Again, these are partitioned according to their defect, this time a non-negative integer $d \equiv N \pmod{2}$, the non-generalized case corresponding to $d = 1$ if N is odd (type B), respectively $d = 0$ if N is even (type D). We denote by $\overline{\mathbf{Symb}}_N^d$ the set of all unordered pairs $\begin{bmatrix} S \\ T \end{bmatrix}$ of finite subsets of $\{0, 1, 2, \dots\}$ such that:

- $|l(S) - l(T)| = d$,

- S and T contain no consecutive integers,
- $\sum_{s \in S} s + \sum_{t \in T} t = \frac{N}{2} + \frac{1}{2}((l(S) + l(T) - 1)^2 - 1)$.

(Note that the last condition implies $l(S) + l(T) \equiv N \pmod{2}$.) Moreover, these pairs are taken modulo the equivalence relation generated by the shift operation:

$$\begin{bmatrix} S \\ T \end{bmatrix} \simeq \begin{bmatrix} \{0\} \cup (S + 2) \\ \{0\} \cup (T + 2) \end{bmatrix}$$

Remark 4.13. If $d > 0$ (which is always the case in type B), we can decide to always write a symbol $\Lambda \in \overline{\mathbf{Symb}}_N^d$ in the form $\begin{bmatrix} S \\ T \end{bmatrix}$ with $l(S) > l(T)$.

We set $\overline{\mathbf{Symb}}_N = \bigcup_{d \equiv N \pmod{2}} \overline{\mathbf{Symb}}_N^d$.

Let us first suppose that N is odd. Let Λ be any symbol in $\overline{\mathbf{Symb}}_N$. Write the entries of Λ in a single row in increasing order:

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq x_{2m}$$

Let $x_0^0 \leq x_1^0 \leq x_2^0 \leq \dots \leq x_{2m}^0$ be the sequence

$$0 \leq 0 \leq 2 \leq 2 \leq \dots \leq 2(m-1) \leq 2(m-1) \leq 2m.$$

Then, we set

$$\delta(\Lambda) := \sum_{0 \leq i < j \leq 2m} \inf(x_i, x_j) - \sum_{0 \leq i < j \leq 2m} \inf(x_i^0, x_j^0).$$

By an easy computation, we get:

$$\delta(\Lambda) = \sum_{0 \leq i < j \leq 2m} \inf(x_i, x_j) - \frac{1}{3}m(m-1)(4m+1).$$

We now suppose that N is even. Let Λ be any symbol in $\overline{\mathbf{Symb}}_N$. Write the entries of Λ in a single row in increasing order:

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq x_{2m-1}$$

Let $x_0^0 \leq x_1^0 \leq x_2^0 \leq \dots \leq x_{2m-1}^0$ be the sequence

$$0 \leq 0 \leq 2 \leq 2 \leq \dots \leq 2(m-1) \leq 2(m-1).$$

Then, we set

$$\delta(\Lambda) := \sum_{0 \leq i < j \leq 2m-1} \inf(x_i, x_j) - \sum_{0 \leq i < j \leq 2m-1} \inf(x_i^0, x_j^0).$$

By an easy computation, we get:

$$\delta(\Lambda) = \sum_{0 \leq i < j \leq 2m-1} \inf(x_i, x_j) - \frac{1}{3}m(m-1)(4m-5).$$

4.2.3. *Parametrization of $\mathfrak{P}_{\mathbb{K}}^G$ when G is a classical group.* We will suppose until the end of the article that $p \neq 2$.

In [Lus84], Lusztig defined a bijection $\mathbf{Symb}_{2n} \leftrightarrow \mathfrak{P}_{\mathbb{K}}^G$ when $G = \mathbf{Sp}_{2n}(\overline{\mathbb{F}}_p)$ and a bijection $\overline{\mathbf{Symb}}_{2n+1} \leftrightarrow \mathfrak{P}_{\mathbb{K}}^G$, when $G = \mathbf{SO}_{2n+1}(\overline{\mathbb{F}}_p)$.

If $G = \mathbf{Sp}_{2n}(\overline{\mathbb{F}}_p)$ or $G = \mathbf{SO}_{2n+1}(\overline{\mathbb{F}}_p)$, let us denote by $(\mathcal{O}(\Lambda), \mathcal{L}(\Lambda)) \in \mathfrak{P}_{\mathbb{K}}^G$ the pair corresponding to a symbol Λ .

A symbol $\Lambda = \begin{pmatrix} S \\ T \end{pmatrix}$ in \mathbf{Symb}_{2n}^1 or in $\overline{\mathbf{Symb}}_{2n+1}^1$ is said to be *distinguished* when

$$s_1 \leq t_1 \leq s_2 \leq t_2 \leq \dots$$

If a symbol Λ is distinguished, then $\mathcal{L}(\Lambda)$ is trivial. Moreover, the map defined by $\Lambda \mapsto \mathcal{O}(\Lambda)$ (see also Subsection 5.3.1) is a bijection between the subset of distinguished symbols in \mathbf{Symb}_{2n}^1 (resp. $\overline{\mathbf{Symb}}_{2n+1}^1$) and nilpotent orbits in the nilpotent cone of $\mathbf{Sp}_{2n}(\overline{\mathbb{F}}_p)$ (resp. $\mathbf{SO}_{2n+1}(\overline{\mathbb{F}}_p)$).

When $G = \mathbf{SO}_{2n}(\overline{\mathbb{F}}_p)$, the pairs $(\mathcal{O}, \mathcal{L})$ in $\mathfrak{P}_{\mathbb{K}}^G$ are in 1-1 correspondence with the set $\overline{\mathbf{Symb}}_{2n}^0$ except that to any symbol $\Lambda = \begin{bmatrix} S \\ S \end{bmatrix}$ in $\overline{\mathbf{Symb}}_{2n}^0$, there correspond two pairs $(\mathcal{O}(\Lambda), 1)$ and $(\mathcal{O}'(\Lambda), 1)$ where $\mathcal{O}(\Lambda)$ and $\mathcal{O}'(\Lambda)$ are two degenerate nilpotent orbits. An element of $\overline{\mathbf{Symb}}_{2n}^0$ of the form $\begin{bmatrix} S \\ S \end{bmatrix}$ is called *degenerate*.

A symbol $\Lambda = \begin{bmatrix} S \\ T \end{bmatrix}$ in $\overline{\mathbf{Symb}}_{2n}^0$ is said to be *distinguished* when

$$s_1 \leq t_1 \leq s_2 \leq t_2 \leq \dots$$

or

$$t_1 \leq s_1 \leq t_2 \leq s_2 \leq \dots$$

We will denote by $(\mathcal{O}(\Lambda), \mathcal{L}(\Lambda))$ the pair in $\mathfrak{P}_{\mathbb{K}}^G$ associated to a non degenerate symbol $\Lambda \in \overline{\mathbf{Symb}}_{2n}^0$. If $\Lambda \in \overline{\mathbf{Symb}}_{2n}^0$ is distinguished, then $\mathcal{L}(\Lambda)$ is trivial. Moreover, the map defined by $\Lambda \mapsto \mathcal{O}(\Lambda)$ (see also Subsection 5.3.1) induces a bijection between the subset of distinguished non degenerate symbols in $\overline{\mathbf{Symb}}_{2n}^0$ and non degenerate nilpotent orbits in the nilpotent cone of $\mathbf{SO}_{2n}(\overline{\mathbb{F}}_p)$.

Let Λ be any symbol in \mathbf{Symb}_{2n} or in $\overline{\mathbf{Symb}}_N$, then we have

$$\delta(\Lambda) = \dim \mathcal{B}_x$$

where $x \in \mathcal{O}(\Lambda)$ (or $x \in \mathcal{O}'(\Lambda)$ when $\Lambda \in \overline{\mathbf{Symb}}_{2n}^0$ is degenerate) and where we denote by \mathcal{B}_x the Springer fiber over x , i.e., the variety of Borel subalgebras of \mathfrak{g} containing x (for a proof, see for instance [GM00], proposition 2.23). Hence, we get:

$$\delta(\Lambda) = \frac{1}{2}(\dim G - \text{rank } G - \dim \mathcal{O}(\Lambda))$$

Definition 4.14 (Order on symbols). We define some partial orders on \mathbf{Symb}_{2n} and on $\overline{\mathbf{Symb}}_N$ by:

$$\Lambda \leq \Lambda' \Leftrightarrow \text{either } \Lambda = \Lambda' \text{ or } \delta(\Lambda) > \delta(\Lambda').$$

Remark 4.15. With the above definition, we get:

$$\mathcal{O}(\Lambda) \leq \mathcal{O}(\Lambda') \Rightarrow \Lambda \leq \Lambda'.$$

4.3. Ordinary Springer correspondence. We will recall Shoji's results about the non-generalized Springer correspondence for classical groups, using Lusztig symbols. We will give the description of $\Phi_{\mathbb{K}}$, since one can recover $\Psi_{\mathbb{K}}$ by tensoring with the sign by Proposition 3.4.

4.3.1. Case $G = \mathbf{Sp}_{2n}(\overline{\mathbb{F}}_p)$. Let W_n be the Weyl group of type C_n . We will identify $\text{Irr } \mathbb{K}W_n$ with \mathfrak{Bip}_n as in Subsection 4.1 and $\mathfrak{P}_{\mathbb{K}}^G$ with \mathbf{Symb}_{2n} as in Subsection 4.2.

The map $\Phi_{\mathbb{K}}^G$ is then described by a bijection from \mathfrak{Bip}_n to \mathbf{Symb}_{2n}^1 that we still denote by $\Phi_{\mathbb{K}}^G$. It can be described as follows.

Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be in \mathfrak{Bip}_n . Here by allowing 0 in the entries of $\lambda^{(1)}$ or $\lambda^{(2)}$, we may write those partitions as

$$\begin{aligned} \lambda^{(1)} &: 0 \leq \lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \dots \leq \lambda_m^{(1)} \leq \lambda_{m+1}^{(1)}, \\ \lambda^{(2)} &: 0 \leq \lambda_1^{(2)} \leq \lambda_2^{(2)} \leq \dots \leq \lambda_m^{(2)} \end{aligned}$$

for some $m \geq 0$. We further assume that $\lambda_1^{(1)} \neq 0$ or $\lambda_1^{(2)} \neq 0$.

We set:

$$(4.1) \quad \begin{cases} S = \lambda_1^{(1)} \leq \lambda_2^{(1)} + 2 \leq \lambda_3^{(1)} + 4 \leq \dots \leq \lambda_{m+1}^{(1)} + 2m \\ T = \lambda_1^{(2)} + 1 \leq \lambda_2^{(2)} + 3 \leq \lambda_3^{(2)} + 5 \leq \dots \leq \lambda_m^{(2)} + 2m - 1 \end{cases}$$

Then we get $\Phi_{\mathbb{K}}^G(\lambda) = \begin{pmatrix} S \\ T \end{pmatrix} \in \mathbf{Symb}_{2n}^1$.

4.3.2. Case $G = \mathbf{SO}_{2n+1}(\overline{\mathbb{F}}_p)$. Let W_n be the Weyl group of type B_n . We will identify $\text{Irr } \mathbb{K}W_n$ with \mathfrak{Bip}_n as in Subsection 4.1 and $\mathfrak{P}_{\mathbb{K}}^G$ with $\overline{\mathbf{Symb}}_{2n+1}$ as in Subsection 4.2.

The map $\Phi_{\mathbb{K}}^G$ is then described by a bijection from \mathfrak{Bip}_n to $\overline{\mathbf{Symb}}_{2n+1}^1$ that we will still denote by $\Phi_{\mathbb{K}}^G$. It can be described as follows.

Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be in \mathfrak{Bip}_n . Here by allowing 0 in the entries of $\lambda^{(1)}$ or $\lambda^{(2)}$, we may write those partitions as

$$\begin{aligned} \lambda^{(1)} &: 0 \leq \lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \dots \leq \lambda_m^{(1)} \leq \lambda_{m+1}^{(1)}, \\ \lambda^{(2)} &: 0 \leq \lambda_1^{(2)} \leq \lambda_2^{(2)} \leq \dots \leq \lambda_m^{(2)} \end{aligned}$$

for some $m \geq 0$ such. We further assume that $\lambda_1^{(1)} \neq 0$ or $\lambda_1^{(2)} \neq 0$.

We set:

$$(4.2) \quad \begin{cases} S = \lambda_1^{(1)} \leq \lambda_2^{(1)} + 2 \leq \lambda_3^{(1)} + 4 \leq \dots \leq \lambda_{m+1}^{(1)} + 2m \\ T = \lambda_1^{(2)} \leq \lambda_2^{(2)} + 2 \leq \lambda_3^{(2)} + 4 \leq \dots \leq \lambda_m^{(2)} + 2m - 2 \end{cases}$$

Then we get $\Phi_{\mathbb{K}}^G(\lambda) = \begin{bmatrix} S \\ T \end{bmatrix} \in \overline{\mathbf{Symb}}_{2n+1}^1$.

4.3.3. *Case $G = \mathbf{SO}_{2n}(\overline{\mathbb{F}}_p)$.* Let W'_n be the Weyl group of type D_n . We have seen in 4.1.2 that we can identify $\text{Irr } \mathbb{K}W'_n$ with $\overline{\mathfrak{Bip}}_n$ except that an element $\lambda = [\lambda^{(1)}, \lambda^{(1)}]$ in $\overline{\mathfrak{Bip}}_n$ corresponds to two irreducible representation $S^{\lambda,+}$ and $S^{\lambda,-}$ of W'_n . Moreover, we have recalled in Subsection 4.2 that $\mathfrak{P}_{\mathbb{K}}^G$ can be identified with $\overline{\mathbf{Symb}}_{2n}$ except that a degenerate symbol $\Lambda = \begin{bmatrix} S \\ S \end{bmatrix}$ corresponds to two pairs $(\mathcal{O}(\Lambda), 1)$ and $(\mathcal{O}'(\Lambda), 1)$.

The map $\Phi_{\mathbb{K}}^G$ is then described by a bijection from $\overline{\mathfrak{Bip}}_n$ to $\overline{\mathbf{Symb}}_{2n}^0$ that we will still denote by $\Phi_{\mathbb{K}}^G$. It can be described as follows.

Let $\lambda = [\lambda^{(1)}, \lambda^{(2)}]$ be in $\overline{\mathfrak{Bip}}_n$. Here by allowing 0 in the entries of $\lambda^{(1)}$ or $\lambda^{(2)}$, we may write those partitions as

$$\begin{aligned} \lambda^{(1)} : 0 \leq \lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \dots \leq \lambda_m^{(1)}, \\ \lambda^{(2)} : 0 \leq \lambda_1^{(2)} \leq \lambda_2^{(2)} \leq \dots \leq \lambda_m^{(2)} \end{aligned}$$

for some $m \geq 0$. We further assume that $\lambda_1^{(1)} \neq 0$ or $\lambda_1^{(2)} \neq 0$.

We set:

$$(4.3) \quad \begin{cases} S = \lambda_1^{(1)} \leq \lambda_2^{(1)} + 2 \leq \lambda_3^{(1)} + 4 \leq \dots \leq \lambda_m^{(1)} + 2m - 2 \\ T = \lambda_1^{(2)} \leq \lambda_2^{(2)} + 2 \leq \lambda_3^{(2)} + 4 \leq \dots \leq \lambda_m^{(2)} + 2m - 2 \end{cases}$$

Then we get $\Phi_{\mathbb{K}}^G(\lambda) = \begin{bmatrix} S \\ T \end{bmatrix} \in \overline{\mathbf{Symb}}_{2n}^0$ and a bipartition $[\lambda^{(1)}, \lambda^{(1)}]$ is sent to a degenerate symbol.

4.4. **Compatibility with Dipper-James order.** Here G is either $\mathbf{Sp}_{2n}(\overline{\mathbb{F}}_p)$ or $\mathbf{SO}_N(\overline{\mathbb{F}}_p)$.

Proposition 4.16. *Let λ and μ be in \mathfrak{Bip}_n (resp. $\overline{\mathfrak{Bip}}_n$) if $G = \mathbf{Sp}_{2n}(\overline{\mathbb{F}}_p)$ or $G = \mathbf{SO}_{2n+1}(\overline{\mathbb{F}}_p)$ (resp. if $G = \mathbf{SO}_{2n}(\overline{\mathbb{F}}_p)$). If we have $\lambda \leq_{DJ} \mu$ and $\lambda \neq \mu$ then we get $\delta(\Phi_{\mathbb{K}}^G(\mu)) < \delta(\Phi_{\mathbb{K}}^G(\lambda))$.*

Proof. The proof is similar for $G = \mathbf{Sp}_{2n}$ and for $G = \mathbf{SO}_N$. We give the proof for $G = \mathbf{Sp}_{2n}$.

Given ν and δ two partitions of length m , it is known that $\delta \leq \nu$ if and only if there is a sequence of partitions $\delta = \delta(0) \leq \delta(1) \leq \dots \leq \delta(r) = \nu$ where for any k there exists $1 \leq a < b \leq m$ such that

$$\delta_a(k) = \delta_a(k+1) + 1, \delta_b(k) = \delta_b(k+1) - 1 \text{ and } \delta_i(k) = \delta_i(k+1) \text{ } i \neq a, b.$$

Similarly, the two bipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ verify $\boldsymbol{\mu} \leq_{DJ} \boldsymbol{\lambda}$ if and only if there exists an increasing sequence

$$\boldsymbol{\mu} = \boldsymbol{\delta}(0) \leq_{DJ} \boldsymbol{\delta}(1) \leq_{DJ} \cdots \leq_{DJ} \boldsymbol{\delta}(r) = \boldsymbol{\lambda}$$

of bipartitions $\boldsymbol{\delta}(k)$ such that $\boldsymbol{\delta}(k)$ is obtained from $\boldsymbol{\delta}(k+1)$ by adding 1 and -1 in two parts of the same partition indexed by integers a, b with $a < b$.

Hence, one can reduce to the situation where $\lambda^{(1)} = \mu^{(1)}$ or $\lambda^{(2)} = \mu^{(2)}$.

For example, let us assume that $\lambda^{(2)} = \mu^{(2)}$ and $\lambda^{(1)} \leq \mu^{(1)}$. Then one can even further reduce to the case when $\mu^{(1)}$ is obtained from $\lambda^{(1)}$ by increasing one part by 1 and by decreasing another by 1.

Hence, if we set $\Phi_{\mathbb{K}}^G(\boldsymbol{\lambda}) = \binom{S}{T} = \begin{pmatrix} s_1 < s_2 < \cdots < s_{m+1} \\ t_1 < t_2 < \cdots < t_m \end{pmatrix}$ and $\Phi_{\mathbb{K}}^G(\boldsymbol{\mu}) = \binom{S'}{T'} = \begin{pmatrix} s'_1 < s'_2 < \cdots < s'_{m+1} \\ t'_1 < t'_2 < \cdots < t'_m \end{pmatrix}$, then $T' = T$ and S' is obtained from S by increasing one entry s_{j_0} by 1 and by decreasing another entry s_{i_0} by 1 (where $i_0 < j_0$).

Let us write the entries of S and T in a single row in increasing order:

$$x : x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{2m}$$

and the entries of S' and T' in the same way:

$$x' : x'_0 \leq x'_1 \leq x'_2 \leq \cdots \leq x'_{2m}.$$

Let i_1 be the smallest integer in $\{0, \dots, 2m\}$ such that $x_{i_1} = s_{i_0}$ and let j_1 be the largest integer in $\{0, \dots, 2m\}$ such that $x_{j_1} = s_{j_0}$. Then, the following properties are verified:

- $i_1 < j_1$,
- $x'_{j_1} = x_{j_1} + 1$,
- $x'_{i_1} = x_{i_1} - 1$,
- $x'_i = x_i$ if $i \neq i_1$ and $i \neq j_1$.

And we get:

$$\begin{aligned} \delta(\Phi_{\mathbb{K}}^G(\boldsymbol{\mu})) - \delta(\Phi_{\mathbb{K}}^G(\boldsymbol{\lambda})) &= \sum_{0 \leq i < j \leq 2m} \inf(x'_i, x'_j) - \sum_{0 \leq i < j \leq 2m} \inf(x_i, x_j) \\ &= \sum_{i=0}^{2m} (2m-i)(x'_i - x_i) \\ &= (2m-i_1)(x'_{i_1} - x_{i_1}) + (2m-j_1)(x'_{j_1} - x_{j_1}) \\ &= i_1 - j_1 < 0 \end{aligned}$$

□

Corollary 4.17. *Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be in \mathfrak{Bip}_n (resp. $\overline{\mathfrak{Bip}}_n$) if $G = \mathbf{Sp}_{2n}(\overline{\mathbb{F}}_p)$ or $G = \mathbf{SO}_{2n+1}(\overline{\mathbb{F}}_p)$ (resp. if $G = \mathbf{SO}_{2n}(\overline{\mathbb{F}}_p)$). If we have $\boldsymbol{\lambda} \leq_{DJ} \boldsymbol{\mu}$ and*

$\lambda \neq \mu$ then we get $\delta(\Psi_{\mathbb{K}}^G(\lambda)) < \delta(\Psi_{\mathbb{K}}^G(\mu))$. Hence,

$$\lambda \leq_{DJ} \mu \Rightarrow \Psi_{\mathbb{K}}^G(\mu) \leq \Psi_{\mathbb{K}}^G(\lambda)$$

for the partial order we have defined on symbols in 4.14.

Proof. The proof uses Proposition 4.16 from assertion $(\mu^* \leq_{DJ} \lambda^*$ and $\mu^* \neq \lambda^*)$. \square

Definition 4.18 (δ -order on bipartitions). Let λ and μ be two bipartitions of n . We say that $\lambda \leq_{\delta} \mu$ if $\Psi_{\mathbb{K}}^G(\mu) \leq \Psi_{\mathbb{K}}^G(\lambda)$.

Corollary 4.19. If $\gamma_{\ell} : \text{Irr } \mathbb{F}W_n \rightarrow \text{Irr } \mathbb{K}W_n$ is defined as in 4.1.1 by

$$\gamma_{\ell}(\mathbf{D}^{(\lambda^{(1)}, \lambda^{(2)})}) := \mathbf{S}^{(\lambda^{(1)}, \lambda^{(2)})}$$

then $(\leq_{\delta}, \gamma_{\ell})$ is a basic set datum.

If $\gamma'_{\ell} : \text{Irr } \mathbb{F}W'_n \rightarrow \text{Irr } \mathbb{K}W'_n$ is defined as in 4.1.2 by:

- $\gamma'_{\ell}(\mathbf{E}^{[\lambda^{(1)}, \lambda^{(2)}]}) := \mathbf{S}^{[\lambda^{(1)}, \lambda^{(2)}]},$
- $\gamma'_{\ell}(\mathbf{E}^{[\lambda, \pm]}) := \mathbf{S}^{[\lambda, \pm]},$

then $(\leq_{\delta}, \gamma'_{\ell})$ is a basic set datum.

Proof. This can be easily verified by using Propositions 2.2, 4.3, 4.9 and Corollary 4.17. \square

Corollary 4.20. The map $\beta_{\ell} : \text{Irr } \mathbb{F}W \rightarrow \text{Irr } \mathbb{K}W$ which was defined in subsection 3.3 is equal to γ_{ℓ} (resp. γ'_{ℓ}) when $W = W_n$ (resp. $W = W'_n$).

Proof. We have seen in Subsection 3.3, that $(\leq_{\text{triv}}^{\mathcal{N}}, \beta_{\ell})$ is a basic set datum. Moreover, by remark 4.15 we have $(\leq_{\text{triv}}^{\mathcal{N}} \subseteq \leq_{\delta})$. By Proposition 2.2, this implies that $(\leq_{\delta}, \beta_{\ell})$ is a basic set datum. Hence, we can conclude by Corollaries 2.4 and 4.19. \square

We can now deduce the main result of this section.

Theorem 4.21. For $\ell \neq 2$, the modular Springer correspondence for classical types is described as follows.

If G is of type B_n or C_n , then for any ℓ -regular bipartition $(\lambda^{(1)}, \lambda^{(2)})$ of n we have

$$\Psi_{\mathbb{F}}^G(\mathbf{D}^{(\lambda^{(1)}, \lambda^{(2)})}) = \Phi_{\mathbb{K}}^G(\mathbf{S}^{(\lambda^{(2)'}), \lambda^{(1)'}})$$

and similarly in type D_n , for any nondegenerate unordered ℓ -regular bipartition $[\lambda^{(1)}, \lambda^{(2)}]$ of n we have

$$\Psi_{\mathbb{F}}^G(\mathbf{D}^{[\lambda^{(1)}, \lambda^{(2)}]}) = \Phi_{\mathbb{K}}^G(\mathbf{S}^{[\lambda^{(2)'}, \lambda^{(1)'}]})$$

while for any degenerate ℓ -regular bipartition $[\lambda, \lambda]$ (in case n is even), we have

$$\Psi_{\mathbb{F}}^G(\mathbf{D}^{[\lambda, \pm]}) = \Phi_{\mathbb{K}}^G(\mathbf{S}^{[\lambda', \pm]})$$

The combinatorial description of $\Phi_{\mathbb{K}}^G$ for each case is recalled in Section 4.3.

5. MORE PRECISE COMBINATORIAL RESULTS

Let G be an algebraic reductive group of classical type over $\overline{\mathbb{F}}_p$ with $p \neq 2$, let W be its Weyl group and let \mathcal{N} be its nilpotent cone.

In Section 4, we used the parametrizations of the set $\mathfrak{P}_{\mathbb{K}}^G$ of pairs $(\mathcal{O}, \mathcal{L})$ by Lusztig's symbols and that of $\text{Irr } \mathbb{K}W$ by a set of bipartitions \mathfrak{Bip}_n or $\overline{\mathfrak{Bip}}_n$. In that context, Springer correspondence was combinatorially described in Subsection 4.3 as an application $\Phi_{\mathbb{K}}$ between bipartitions and symbols.

However, it is well known that the nilpotent orbits of a group of classical type can be labeled by some partitions. It is then interesting to describe Springer correspondence in terms of partitions instead of symbols.

Let us denote by \mathcal{O}_{λ} the nilpotent orbit in \mathcal{N} parametrized by a partition λ . Then, a theorem, essentially proved by Gerstenhaber in [Ger61b] and [Ger61a], says that $\mathcal{O}_{\lambda} < \mathcal{O}_{\mu}$ if and only if $\lambda < \mu$ for the dominance order on partitions. At the end of this section, we prove that the Dipper-James order on \mathfrak{Bip}_n and $\overline{\mathfrak{Bip}}_n$ is compatible, via Springer correspondence, with the dominance order on partitions. This result gives an alternative proof of Corollary 4.20.

5.1. Some explicit formulas for the 2-quotient. If G is of type $X_n \in \{C_n, B_n, D_n\}$, we will denote by $\mathcal{P}(X_n)$ the set of partitions parametrizing the nilpotent orbits of \mathcal{N} . Springer correspondence induces an injection from the set of nilpotent orbits of \mathcal{N} to $\text{Irr } \mathbb{K}W$. This injection will be combinatorially described in this Subsection as an injection from $\mathcal{P}(X_n)$ to \mathfrak{Bip}_n or $\overline{\mathfrak{Bip}}_n$.

- (1) $\mathcal{P}(C_n)$ is the set of partitions of $2n$ in which each odd part appears with an even multiplicity,
- (2) $\mathcal{P}(B_n)$ is the set of partitions of $2n+1$ in which each even part appears with an even multiplicity,
- (3) $\mathcal{P}(D_n)$ is the set of partitions of $2n$ in which each even part appears with an even multiplicity, but in this case, a partition which contains no odd parts corresponds to two degenerate nilpotent orbits.

By adding parts equal to 0 if necessary, we shall assume following [Car93] that the partitions of $\mathcal{P}(C_n) \cup \mathcal{P}(D_n)$ and $\mathcal{P}(B_n)$ have even and odd lengths, respectively. For a partition λ , we denote by \mathcal{O}_{λ} the associated nilpotent orbit.

We refer to [Mac98] for the definition of the 2-quotient $\Delta(\lambda) = (\lambda^{(1)}, \lambda^{(2)})$ of the partition λ . In the case $\lambda \in \mathcal{P}(B_n) \cup \mathcal{P}(C_n) \cup \mathcal{P}(D_n)$ considered here, there exist some explicit formulas for $\lambda^{(1)}$ and $\lambda^{(2)}$.

- (1) For λ in $\mathcal{P}(C_n)$, we write $\{\lambda_{i_1}, \lambda_{i_1+1}, \lambda_{i_2}, \lambda_{i_2+1}, \dots\}$ for the set of odd entries in λ , with $i_1 \ll i_2 \ll \dots$, then we set $I = \{i_1, i_2, \dots, i_a\}$.
- (2) For λ in $\mathcal{P}(B_n)$ or in $\mathcal{P}(D_n)$, we write $\{\lambda_{i_1}, \lambda_{i_1+1}, \lambda_{i_2}, \lambda_{i_2+1}, \dots\}$ for the set of even entries in λ , with $i_1 \ll i_2 \ll \dots$, then we set $I = \{i_1, i_2, \dots, i_a\}$.

For any $s = 1, \dots, a-1$, we have $i_s - i_{s+1} > 1$. The following lemma is proved by using the inverse 2-quotient algorithm.

Lemma 5.1.

- (1) If λ is in $\mathcal{P}(B_n) \cup \mathcal{P}(C_n) \cup \mathcal{P}(D_n)$, then $\Delta(\lambda) = (\lambda^{(1)}, \lambda^{(2)})$ is a bipartition of n .
- (2) When $\lambda \in \mathcal{P}(C_n)$, $\lambda^{(1)}$ and $\lambda^{(2)}$ have length m and for any $i = 1, \dots, m$

$$\lambda_i^{(1)} = \begin{cases} \frac{\lambda_{2i}}{2} & \text{if } \lambda_{2i} \text{ is even,} \\ \frac{\lambda_{2i+1}}{2} & \text{if } 2i \in I \text{ (thus } \lambda_{2i} \text{ odd),} \\ \frac{\lambda_{2i-1}}{2} & \text{if } 2i-1 \in I \text{ (thus } \lambda_{2i} \text{ odd),} \end{cases}$$

$$\lambda_i^{(2)} = \begin{cases} \frac{\lambda_{2i-1}}{2} & \text{if } \lambda_{2i-1} \text{ is even,} \\ \frac{\lambda_{2i-1+1}}{2} & \text{if } 2i-1 \in I \text{ (thus } \lambda_{2i-1} \text{ odd),} \\ \frac{\lambda_{2i-1-1}}{2} & \text{if } 2i-2 \in I \text{ (thus } \lambda_{2i-1} \text{ odd).} \end{cases}$$

- (3) When $\lambda \in \mathcal{P}(B_n)$, $\lambda^{(1)}$ and $\lambda^{(2)}$ have length $m+1$ and m , respectively. For any $i = 1, \dots, m$

$$\lambda_i^{(1)} = \begin{cases} \frac{\lambda_{2i-1}-1}{2} & \text{if } \lambda_{2i-1} \text{ is odd,} \\ \frac{\lambda_{2i-1}}{2} & \text{if } 2i-1 \in I \text{ (thus } \lambda_{2i-1} \text{ even),} \\ \frac{\lambda_{2i-1}}{2} - 1 & \text{if } 2i-2 \in I \text{ (thus } \lambda_{2i-1} \text{ even).} \end{cases}$$

$$\lambda_i^{(2)} = \begin{cases} \frac{\lambda_{2i+1}}{2} & \text{if } \lambda_{2i} \text{ is odd,} \\ \frac{\lambda_{2i}}{2} + 1 & \text{if } 2i \in I \text{ (thus } \lambda_{2i} \text{ even),} \\ \frac{\lambda_{2i}}{2} & \text{if } 2i-1 \in I \text{ (thus } \lambda_{2i} \text{ even),} \end{cases}$$

- (4) When $\lambda \in \mathcal{P}(D_n)$, $\lambda^{(1)}$ and $\lambda^{(2)}$ have length m and for any $i = 1, \dots, m$

$$\lambda_i^{(1)} = \begin{cases} \frac{\lambda_{2i-1}-1}{2} & \text{if } \lambda_{2i-1} \text{ is odd,} \\ \frac{\lambda_{2i-1}}{2} & \text{if } 2i-1 \in I \text{ (thus } \lambda_{2i-1} \text{ even),} \\ \frac{\lambda_{2i-1}}{2} - 1 & \text{if } 2i-2 \in I \text{ (thus } \lambda_{2i-1} \text{ even),} \end{cases}$$

$$\lambda_i^{(2)} = \begin{cases} \frac{\lambda_{2i}+1}{2} & \text{if } \lambda_{2i} \text{ is odd,} \\ \frac{\lambda_{2i}}{2} + 1 & \text{if } 2i \in I \text{ (thus } \lambda_{2i} \text{ even),} \\ \frac{\lambda_{2i}}{2} & \text{if } 2i-1 \in I \text{ (thus } \lambda_{2i} \text{ even)} \end{cases}$$

as in type B_n .

Note that such simple explicit formulas do not exist in general for the ℓ -quotient of a partition. The case we consider here is very particular for we have $\lambda \in \mathcal{P}(B_n) \cup \mathcal{P}(C_n) \cup \mathcal{P}(D_n)$ and $\ell = 2$.

Example 5.2.

- (1) Consider $\lambda = (2, 3, 3, 4, 6, 6, 7, 7, 9, 9)$. Then $I = \{2, 7, 9\}$. We obtain

$$\lambda^{(1)} = \left(\frac{3+1}{2}, \frac{4}{2}, \frac{6}{2}, \frac{7-1}{2}, \frac{9-1}{2} \right) = (2, 2, 3, 3, 4),$$

$$\lambda^{(2)} = \left(\frac{2}{2}, \frac{3-1}{2}, \frac{6}{2}, \frac{7+1}{2}, \frac{9+1}{2} \right) = (1, 1, 3, 4, 5).$$

- (2) Consider $\lambda = (1, 2, 2, 3, 4, 4, 6, 6, 7, 8, 8)$. Then $I = \{2, 5, 7, 10\}$. We obtain

$$\lambda^{(1)} = \left(\frac{1-1}{2}, \frac{2}{2} - 1, \frac{4}{2}, \frac{6}{2}, \frac{7-1}{2}, \frac{8}{2} - 1 \right) = (0, 0, 2, 3, 3, 3),$$

$$\lambda^{(2)} = \left(\frac{2}{2} + 1, \frac{3+1}{2}, \frac{4}{2}, \frac{6}{2}, \frac{8}{2} + 1 \right) = (2, 2, 2, 3, 5).$$

- (3) Consider $\lambda = (2, 2, 3, 4, 4, 6, 6, 7, 8, 8)$. Then $I = \{1, 4, 6, 9\}$. We obtain

$$\lambda^{(1)} = \left(\frac{2}{2}, \frac{3-1}{2}, \frac{4}{2} - 1, \frac{6}{2} - 1, \frac{8}{2} \right) = (1, 1, 1, 2, 4),$$

$$\lambda^{(2)} = \left(\frac{2}{2}, \frac{4}{2} + 1, \frac{6}{2} + 1, \frac{7+1}{2}, \frac{8}{2} \right) = (1, 3, 4, 4, 4).$$

For any $X_n \in \{B_n, C_n, D_n\}$, set

$$\mathfrak{Bip}_\Delta(X_n) = \Delta(\mathcal{P}(X_n)).$$

5.2. From bipartitions to symbols. Springer Correspondence is an injection from $\text{Irr } \mathbb{K}W$ to $\mathfrak{P}_{\mathbb{K}}^G$. For any module $E \in \text{Irr } \mathbb{K}W$, let us denote $(\mathcal{O}_E, \mathcal{L}_E)$ the corresponding pair in $\mathfrak{P}_{\mathbb{K}}^G$. In Subsections 5.2 and 5.3, we describe combinatorially the map from $\text{Irr } \mathbb{K}W$ to the set of nilpotent orbits of \mathcal{N} , defined by $E \mapsto \mathcal{O}_E$. Let us recall that $\text{Irr } \mathbb{K}W$ is parametrized by a set of bipartitions \mathfrak{Bip}_n or $\overline{\mathfrak{Bip}}_n$ and nilpotent orbits of \mathcal{N} are labeled by a set of partition $\mathcal{P}(X_n)$. In Subsection 5.2, we recall how to get a symbol from a bipartition and in Subsection 5.3 we explain how to get a partition from a symbol.

Consider $(\lambda^{(1)}, \lambda^{(2)}) \in \mathfrak{Bip}_{\Delta}(X_n)$. We renormalize $(\lambda^{(1)}, \lambda^{(2)})$ so that $\lambda^{(1)}$ has one part more than $\lambda^{(2)}$ for $X_n = B_n, D_n$ and $\lambda^{(1)}, \lambda^{(2)}$ have the same number of parts for $X_n = D_n$. This is to make the notation consistent with [Car93]. The symbol $\begin{pmatrix} S \\ T \end{pmatrix}$ associated to $(\lambda^{(1)}, \lambda^{(2)})$ (resp. associated to $[\lambda^{(1)}, \lambda^{(2)}]$) was introduced in §4.2

$$(5.1) \quad \begin{aligned} \begin{pmatrix} S \\ T \end{pmatrix} &= \begin{pmatrix} 0 & \lambda_1^{(1)} + 2 & \lambda_2^{(1)} + 4 & \lambda_3^{(1)} + 6 \cdots \\ \lambda_1^{(2)} + 1 & \lambda_2^{(2)} + 3 & \lambda_2^{(2)} + 5 \cdots \end{pmatrix} \in \mathbf{Symb}_{2n}^1 \text{ for } X_n = C_n, \\ \begin{bmatrix} S \\ T \end{bmatrix} &= \begin{bmatrix} \lambda_1^{(1)} & \lambda_2^{(1)} + 2 & \lambda_3^{(1)} + 4 \cdots \\ \lambda_1^{(2)} & \lambda_2^{(2)} + 2 \cdots \end{bmatrix} \in \overline{\mathbf{Symb}}_{2n+1}^1 \text{ for } X_n = B_n, \\ \begin{bmatrix} S \\ T \end{bmatrix} &= \begin{bmatrix} \lambda_1^{(1)} & \lambda_2^{(1)} + 2 & \lambda_3^{(1)} + 4 \cdots \\ \lambda_1^{(2)} & \lambda_2^{(2)} + 2 & \lambda_3^{(1)} + 4 \cdots \end{bmatrix} \in \overline{\mathbf{Symb}}_{2n}^0 \text{ for } X_n = D_n. \end{aligned}$$

We define the *reading* of the symbol $\begin{pmatrix} S \\ T \end{pmatrix}$ as the sequence

$$w\left(\begin{pmatrix} S \\ T \end{pmatrix}\right) = (s_1, t_1, s_2, t_2, s_3, \cdots)$$

obtained by considering alternatively the entries of S and T . So, for $X_n = B_n, C_n$, a symbol $\begin{pmatrix} S \\ T \end{pmatrix}$ is *distinguished* (see Subsection 4.2) when the entries of its reading weakly increase from left to right. For $X_n = D_n$, a symbol $\begin{bmatrix} S \\ T \end{bmatrix}$ is distinguished if the letters of $w\left(\begin{pmatrix} S \\ T \end{pmatrix}\right)$ or $w\left(\begin{pmatrix} T \\ S \end{pmatrix}\right)$ weakly increase from left to right.

From Lemma 5.1 and by using the fact that the parts of λ increase from left to right, we deduce the following characterization of $\mathfrak{Bip}_{\Delta}(X_n)$.

Lemma 5.3.

- For $X_n = B_n, C_n$, $(\lambda^{(1)}, \lambda^{(2)}) \in \mathfrak{Bip}_{\Delta}(X_n)$ if and only its symbol is distinguished.

- For $X_n = D_n$, $(\lambda^{(1)}, \lambda^{(2)}) \in \mathfrak{Bip}_\Delta(D_n)$ if and only the symbol corresponding to $[\lambda^{(1)}, \lambda^{(2)}]$ is distinguished where $[\lambda^{(1)}, \lambda^{(2)}]$ is the unordered bipartition associated to $(\lambda^{(1)}, \lambda^{(2)})$.

Example 5.4. For the bipartition appearing in the previous example we have

$$\begin{aligned}
 (1) \quad & \begin{pmatrix} S \\ T \end{pmatrix} = \begin{pmatrix} 0 & 4 & 6 & 9 & 11 & 14 \\ 2 & 4 & 8 & 11 & 14 \end{pmatrix}. \\
 (2) \quad & \begin{pmatrix} S \\ T \end{pmatrix} = \begin{pmatrix} 0 & 2 & 6 & 9 & 11 & 13 \\ 2 & 4 & 6 & 9 & 13 \end{pmatrix}. \\
 (3) \quad & \begin{bmatrix} S \\ T \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 & 8 & 12 \\ 1 & 5 & 8 & 10 & 12 \end{bmatrix}.
 \end{aligned}$$

5.3. Recovering a partition from its symbol.

5.3.1. *When the symbol is distinguished.* Given a bipartition $(\lambda^{(1)}, \lambda^{(2)}) \in \mathfrak{Bip}_\Delta(X_n)$, we now consider the problem of finding the unique $\lambda \in \mathcal{P}(X_n)$ such that $\Delta(\lambda) = (\lambda^{(1)}, \lambda^{(2)})$. Set Λ be the symbol associated to $(\lambda^{(1)}, \lambda^{(2)})$ for $X_n = B_n, C_n$ (resp. $[\lambda^{(1)}, \lambda^{(2)}]$ for $X_n = D_n$) as in (5.1) and write $\psi = w(\Lambda)$ the reading of Λ . The letters of ψ weakly increase from left to right.

Assume first $X_n = C_n$. By a distinguished symbol $\Lambda = \begin{pmatrix} S \\ T \end{pmatrix}$, we mean a pair (s_i, t_i) with $s_i = t_i$ or a pair (t_i, s_{i+1}) with $t_i = s_{i+1}$. Since the rows of the symbols strictly increase, the distinguished pairs are pairwise disjoint. Let $\tilde{\psi}$ be the sequence obtained from ψ by changing each distinguished pair (s_i, s_i) into $(s_i - \frac{1}{2}, s_i + \frac{1}{2})$. Then $\lambda = 2(\tilde{\psi} - \rho_{2m+1})$ where $\rho_{2m+1} = (0, 1, 2, \dots, 2m)$.

Assume now, $X_n = B_n$ or D_n . A distinguished pair is a pair (t_i, s_{i+1}) such that $t_i = s_{i+1}$. A frozen pair is a pair (s_i, t_i) with $s_i = t_i$. Since the rows of the symbols strictly increase, distinguished and frozen pairs are disjoint. Let $\tilde{\psi}$ be the sequence obtained from ψ by changing each distinguished pair (t_i, t_i) into $(t_i - 1, t_i + 1)$, each s_i into $s_i + \frac{1}{2}$ and each t_i into $t_i - \frac{1}{2}$ provided they do not appear in a frozen pair. Then $\lambda = 2(\tilde{\psi} - \rho'_{2m+1})$ where $\rho'_{2m+1} = (0, 0, 2, 2, \dots, 2m, 2m, 2m+1) \in \mathbb{Z}^{2m+1}$ for $X_n = B_n$ and $\lambda = 2(\tilde{\psi} - \rho'_{2m})$ where $\rho'_{2m} = (0, 0, 2, 2, \dots, 2m, 2m) \in \mathbb{Z}^{2m}$ for $X_n = D_n$.

Example 5.5. One verifies that the above procedure applied to the symbols obtained in Example 5.4 yield the partitions introduced in Example 5.2.

We have thus the following Proposition:

Proposition 5.6. *The map Δ yields a bijection between $\mathcal{P}(X_n)$ and $\mathfrak{Bip}_\Delta(X_n)$.*

5.3.2. *When the symbol is not distinguished.* For any $s \in \mathbb{N}^l$ and any $i \in \{1, \dots, l\}$ set

$$S_i(s) = s_i + s_{i+1} + \dots + s_{2l}.$$

We denote by \leq the partial order defined on \mathbb{N}^l such that

$$(5.2) \quad s \leq s' \iff S_i(s) \leq S_i(s') \text{ for any } i = 1, \dots, 2l.$$

It naturally extends the dominance order on partitions.

We now recall the procedure associating to a symbol Λ of any set \mathbf{Symb}_{2n}^1 , $\overline{\mathbf{Symb}}_{2n+1}^1$ or $\overline{\mathbf{Symb}}_{2n}^0$, a partition λ giving the “unipotent class part” C of the associated pair (C, \mathcal{L}) . It can be decomposed in two steps:

- (1) First write the entries of Λ in increasing order. This gives the reading of a distinguished symbol, says $\hat{\Lambda}$.
- (2) The partition λ is then the unique partition such that the symbol of $\Delta(\lambda)$ coincides with $\hat{\Lambda}$.

It is not a priori totally immediate that in step 1, the reordering of the entries of Λ yields the reading of a symbol in \mathbf{Symb}_{2n}^1 , in $\overline{\mathbf{Symb}}_{2n+1}^1$, or in $\overline{\mathbf{Symb}}_{2n}^0$ (i.e. corresponding to a bipartition). This follows from the straightening procedure below.

Let Y be the set of pairs (s_i, t_i) or (t_i, s_{i+1}) in a symbol $\binom{S}{T}$ such that $s_i > t_i$ or $t_i > s_{i+1}$, respectively. Since the rows of $\binom{S}{T}$ strictly increase, an entry of $\binom{S}{T}$ cannot appear in more than one pair of the set Y . Now, by changing in $\binom{S}{T}$ the pairs (s_i, t_i) and (t_i, s_{i+1}) of Y into (t_i, s_i) and (s_{i+1}, t_i) , we yet obtain the symbol of a bipartition, says $\binom{S^{(1)}}{T^{(1)}}$. Moreover if we set $\psi = w(\binom{S}{T})$ and $\psi^{(1)} = w(\binom{S^{(1)}}{T^{(1)}})$, we have $\psi^{(1)} > \psi$ for the dominance order (5.2) when $Y \neq \emptyset$. Iterations of this procedure yield the desired distinguished symbol $\binom{\hat{S}}{\hat{T}}$.

Example 5.7. For $X_n = C_{31}$ and $(\lambda^{(1)}, \lambda^{(2)}) = ((1, 4, 4, 5, 6, 7), (1, 3))$, we have

$$\binom{S}{T} = \begin{pmatrix} 1 & 6 & 8 & 11 & 14 & 17 \\ & 1 & 3 & 5 & 8 & 12 \end{pmatrix} \text{ and } Y = \{(6, 3), (8, 5), (11, 8), (14, 12)\}.$$

This gives by the previous procedure

$$\binom{S^{(1)}}{T^{(1)}} = \begin{pmatrix} 1 & 3 & 5 & 8 & 12 & 17 \\ & 1 & 6 & 8 & 11 & 14 \end{pmatrix} \text{ and } Y^{(1)} = \{(6, 5)\}.$$

Then

$$\begin{pmatrix} S^{(2)} \\ T^{(2)} \end{pmatrix} = \begin{pmatrix} \hat{S} \\ \hat{T} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 & 8 & 12 & 17 \\ & 1 & 5 & 8 & 11 & 14 \end{pmatrix}$$

is distinguished. With the notation of the previous paragraph, the distinguished pairs in $\begin{pmatrix} \hat{S} \\ \hat{T} \end{pmatrix}$ are $(1, 1)$ and $(8, 8)$. We have

$$\begin{aligned} \psi &= (1, 1, 3, 5, 6, 8, 8, 11, 12, 14, 17), \\ \tilde{\psi} &= \left(\frac{1}{2}, \frac{3}{2}, 3, 5, 6, \frac{15}{2}, \frac{17}{2}, 11, 12, 14, 17 \right). \end{aligned}$$

This finally gives

$$\lambda = 2 \left(\frac{1}{2}, \frac{1}{2}, 1, 2, 2, \frac{5}{2}, \frac{5}{2}, 4, 4, 5, 7 \right) = (1, 1, 2, 4, 4, 5, 5, 8, 8, 10, 14).$$

5.4. Compatibility between Springer and Dipper-James orders. Consider $(\lambda^{(1)}, \lambda^{(2)})$ and $(\mu^{(1)}, \mu^{(2)})$ two bipartitions. We recall the Dipper-James order on (ordered) bipartitions $(\mu^{(1)}, \mu^{(2)}) \leq_{DJ} (\lambda^{(1)}, \lambda^{(2)})$ if and only if $\mu^{(1)} \leq \lambda^{(1)}$ and $\mu^{(2)} \leq \lambda^{(2)}$ where \leq is the dominance order (5.2). Denote by $\begin{pmatrix} S \\ T \end{pmatrix}$ (resp. $\begin{pmatrix} S' \\ T' \end{pmatrix}$) the symbol associated to $(\lambda^{(1)}, \lambda^{(2)})$ (resp. to $(\mu^{(1)}, \mu^{(2)})$). Then,

$$(5.3) \quad (\mu^{(1)}, \mu^{(2)}) \leq_{DJ} (\lambda^{(1)}, \lambda^{(2)}) \iff S \leq S' \text{ and } T \leq T'.$$

Proposition 5.8. Assume $G = \mathbf{SO}_{2n+1}$ or \mathbf{Sp}_{2n} . Consider $(\lambda^{(1)}, \lambda^{(2)})$ and $(\mu^{(1)}, \mu^{(2)})$ in \mathfrak{Bip}_n . Let λ (resp. μ) be the partition obtained from $(\lambda^{(1)}, \lambda^{(2)})$ (resp. $(\mu^{(1)}, \mu^{(2)})$) after the procedures described in subsections 5.2 and 5.3. Then

$$(\mu^{(1)}, \mu^{(2)}) \leq_{DJ} (\lambda^{(1)}, \lambda^{(2)}) \implies \mu \leq \lambda.$$

Proof. By the previous arguments, we can assume that $(\mu^{(1)}, \mu^{(2)})$ is obtained from $(\lambda^{(1)}, \lambda^{(2)})$ by adding 1 and -1 in two distinct parts of the same partition. Let us write $\begin{pmatrix} S' \\ T' \end{pmatrix}$ and $\begin{pmatrix} S \\ T \end{pmatrix}$ for the symbols of the bipartitions $(\mu^{(1)}, \mu^{(2)})$ and $(\lambda^{(1)}, \lambda^{(2)})$. These two symbols coincide up to the change of a pair $(x, y), x < y$ of entries appearing in the same row of $\begin{pmatrix} S' \\ T' \end{pmatrix}$ into $(x+1, y-1)$. Define then the symbols $\begin{pmatrix} \hat{S}' \\ \hat{T}' \end{pmatrix}$ and $\begin{pmatrix} \hat{S} \\ \hat{T} \end{pmatrix}$ by reordering $\begin{pmatrix} S' \\ T' \end{pmatrix}$ and $\begin{pmatrix} S \\ T \end{pmatrix}$ as in § 5.3.2. Set $\psi = w(\begin{pmatrix} \hat{S} \\ \hat{T} \end{pmatrix})$ and $\psi' = w(\begin{pmatrix} \hat{S}' \\ \hat{T}' \end{pmatrix})$. Then ψ' is obtained from ψ by modifying a pair of entries (x, y) into $(x+1, y-1)$. Since the entries of ψ and ψ' increase from left to right there exist $a < b$ such that $(\psi_a, \psi_b) = (x, y)$ and $(\psi'_a, \psi'_b) = (x+1, y-1)$. Observe that $\psi' \leq \psi$. Let $\tilde{\psi}$ and $\tilde{\psi}'$ be the sequence obtained from $\begin{pmatrix} \hat{S} \\ \hat{T} \end{pmatrix}$ and $\begin{pmatrix} \hat{S}' \\ \hat{T}' \end{pmatrix}$ as in § 5.3.1. We have to prove that $\tilde{\psi}' \leq \tilde{\psi}$.

Set l for the number of entries in $\psi, \psi', \tilde{\psi}, \tilde{\psi}'$. Assume $G = \mathbf{Sp}_{2n}$. For any $i \in \{1, \dots, l\}$, we have

$$(5.4) \quad \begin{aligned} S_i(\psi') &= \begin{cases} S_i(\psi) & \text{for } i \notin]a, b] \\ S_i(\psi) - 1 & \text{for } i \in]a, b] \end{cases} , \\ S_i(\tilde{\psi}) &= \begin{cases} S_i(\psi) & \text{if } \psi_{i-1} < \psi_i \\ S_i(\psi) + \frac{1}{2} & \text{if } \psi_{i-1} = \psi_i \end{cases} \quad S_i(\tilde{\psi}') = \begin{cases} S_i(\psi') & \text{if } \psi'_{i-1} < \psi'_i \\ S_i(\psi') + \frac{1}{2} & \text{if } \psi'_{i-1} = \psi'_i \end{cases} . \end{aligned}$$

This can be rewritten on the form

$$\begin{aligned} S_i(\psi') &= S_i(\psi) - \mathbf{1}_{]a, b]}(i), \\ S_i(\tilde{\psi}) &= S_i(\psi) + \frac{1}{2} \delta_{\psi_{i-1}, \psi_i}, \quad S_i(\tilde{\psi}') = S_i(\psi') + \frac{1}{2} \delta_{\psi'_{i-1}, \psi'_i} \end{aligned}$$

where

$$\mathbf{1}_{]a, b]}(i) = \begin{cases} 1 & \text{if } a < i \leq b \\ 0 & \text{otherwise} \end{cases}$$

and δ is the usual Kronecker symbol. Finally, we obtain

$$S_i(\tilde{\psi}') = S_i(\tilde{\psi}) - \mathbf{1}_{]a, b]}(i) + \frac{1}{2} (\delta_{\psi'_{i-1}, \psi'_i} - \delta_{\psi_{i-1}, \psi_i}).$$

We claim that $S_i(\tilde{\psi}') \leq S_i(\tilde{\psi})$. Otherwise, $i \notin]a, b]$, $\delta_{\psi'_{i-1}, \psi'_i} = 1$ and $\delta_{\psi_{i-1}, \psi_i} = 0$. Thus we must have $i = a$ and $\psi_{a-1} = \psi_a + 1$, or $i = b + 1$ and $\psi_b - 1 = \psi_{b+1}$. In both cases, we derive a contradiction for the entries of ψ weakly increase from left to right. Since $S_i(\tilde{\psi}') \leq S_i(\tilde{\psi})$ for any $i = 1, \dots, l$, $\tilde{\psi}' \leq \tilde{\psi}$ and we are done.

Now, assume $G = \mathbf{SO}_{2n+1}$. Then $l = 2m + 1$. According to § 5.3.1, we can form disjoint pairs (ψ_{i-1}, ψ_i) of consecutive letters in ψ such that $\psi_{i-1} = \psi_i$. For such a pair we have

$$(\psi'_{i-1}, \psi'_i) = \begin{cases} (\psi_{i-1}, \psi_i) & \text{if } \psi_{i-1} = \psi_i \text{ and } i \text{ is even,} \\ (\psi_{i-1} - 1, \psi_i + 1) & \text{if } \psi_{i-1} = \psi_i \text{ and } i \text{ is odd.} \end{cases}$$

Let K be the set of indices i such that ψ_i belongs to one of these pairs. Now for any $i \notin K$

$$\psi'_i = \begin{cases} \psi_i - \frac{1}{2} & \text{if } i \text{ is even,} \\ \psi_i + \frac{1}{2} & \text{if } i \text{ is odd.} \end{cases}$$

Consider $i \in \{1, \dots, l\}$. When i is even, (ψ_{i-1}, ψ_i) is not a distinguished pair. Moreover, the set $U_i = \{j \in \{i, \dots, 2m+1\} \mid j \notin K\}$ contains the same number of even and odd indices. Thus $S_i(\psi) = S_i(\psi)$. When i is

odd, by computing the number of odd and even indices in U_i in both cases (ψ_{i-1}, ψ_i) distinguished or not, we obtain

$$S_i(\tilde{\psi}) = S_i(\psi) + \frac{1}{2}\delta_{\psi_{i-1}, \psi_i}(i) + \frac{1}{2}.$$

We have the same relation between $S_i(\tilde{\psi}')$ and $S_i(\psi')$. Moreover $S_i(\psi') = S_i(\psi) - \mathbf{1}_{[a,b]}(i)$. This gives

$$S_i(\tilde{\psi}') = \begin{cases} S_i(\tilde{\psi}) - \mathbf{1}_{[a,b]}(i) & \text{if } i \text{ is even,} \\ S_i(\tilde{\psi}) - \mathbf{1}_{[a,b]}(i) + \frac{1}{2}(\delta_{\psi'_{i-1}, \psi'_i} - \delta_{\psi_{i-1}, \psi_i}) & \text{if } i \text{ is odd.} \end{cases}$$

When i is even, we have clearly $S_i(\tilde{\psi}') \leq S_i(\tilde{\psi})$. This is also true for i odd by using the same arguments as in type C_n . \square

Proposition 5.9. *Assume $G = \mathbf{SO}_{2n}$. Consider $[\lambda^{(1)}, \lambda^{(2)}]$ and $[\mu^{(1)}, \mu^{(2)}]$ in $\overline{\mathfrak{Bip}}_n$. Let λ (resp. μ) be the partition obtained from $[\lambda^{(1)}, \lambda^{(2)}]$ (resp. $[\mu^{(1)}, \mu^{(2)}]$) after the procedures described in subsections 5.2 and 5.3. Then*

$$[\mu^{(1)}, \mu^{(2)}] \leq_{DJ} [\lambda^{(1)}, \lambda^{(2)}] \implies \mu \leq \lambda.$$

Proof. By definition of the Dipper-James order on unordered bipartitions, we can assume (up to a flip of $\lambda^{(1)}$ and $\lambda^{(2)}$) that $(\mu^{(1)}, \mu^{(2)}) \leq_{DJ} (\lambda^{(1)}, \lambda^{(2)})$. We then proceed as in the case $G = \mathbf{SO}_{2n+1}$ of the previous proof by switching the cases i even and i odd due to the fact that $l = 2m$ is even. This also gives $S_i(\tilde{\psi}') \leq S_i(\tilde{\psi})$ for any $i = 1, \dots, 2m$. \square

As a consequence, we obtain another independent proof of Corollary 4.20.

Corollary 5.10. *The map $\beta_\ell : \text{Irr } \mathbb{F}W \rightarrow \text{Irr } \mathbb{K}W$ which was introduced in Subsection 3.3 is defined from the inclusion $\mathfrak{Bip}_n^\ell \hookrightarrow \mathfrak{Bip}_n$ in types B and C (resp. $\overline{\mathfrak{Bip}}_n^\ell \hookrightarrow \overline{\mathfrak{Bip}}_n$ in type D).*

Proof. For the groups of classical types in characteristic $p \neq 2$, the natural order on nilpotent classes coincides with the dominance order on partitions. By the two previous propositions, this yields the inclusion $\leq_{DJ} \subseteq \leq_{\text{triv}}^N$. We then conclude by using Proposition 4.3 and Corollary 2.4. \square

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